

Exponentially and non-exponentially small splitting of separatrices for the pendulum with a fast meromorphic perturbation

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2012 Nonlinearity 25 1367

(<http://iopscience.iop.org/0951-7715/25/5/1367>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 128.112.202.165

The article was downloaded on 12/04/2012 at 18:30

Please note that [terms and conditions apply](#).

Exponentially and non-exponentially small splitting of separatrices for the pendulum with a fast meromorphic perturbation

Marcel Guardia¹ and Tere M Seara²

¹ Department of Mathematics, Mathematics Building, University of Maryland, College Park, MD 20742-4015, USA

² Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

E-mail: marcel.guardia@upc.edu and tere.m-seara@upc.edu

Received 1 August 2011, in final form 26 January 2012

Published 10 April 2012

Online at stacks.iop.org/Non/25/1367

Recommended by D Treschev

Abstract

In this paper, we study the splitting of separatrices phenomenon which arises when one considers a Hamiltonian system of one degree of freedom with a fast periodic or quasiperiodic and meromorphic in the state variables perturbation. The obtained results are different from the previous ones in the literature, which mainly assume algebraic or trigonometric polynomial dependence on the state variables. As a model, we consider the pendulum equation with several meromorphic perturbations and we show the sensitivity of the size of the splitting on the width of the analyticity strip of the perturbation with respect to the state variables. We show that the size of the splitting is exponentially small if the strip of analyticity is wide enough. Furthermore, we see that the splitting grows as the width of the analyticity strip shrinks, even becoming non-exponentially small for very narrow strips. Our results prevent use of polynomial truncations of the meromorphic perturbation to compute the size of the splitting of separatrices.

Mathematics Subject Classification: 34C29, 34C37, 37C29, 34E10

1. Introduction

Exponentially small splitting of separatrices appears in analytic dynamical systems with different time scales. A paradigmatic example is analytic Hamiltonian systems of one degree of

freedom with a fast non-autonomous periodic or quasiperiodic perturbation. Namely, systems of the form

$$\begin{aligned} H\left(x, y, \frac{t}{\varepsilon}\right) &= H_0(x, y) + \mu\varepsilon^\eta H_1\left(x, y, \frac{t}{\varepsilon}\right) \\ &= \frac{y^2}{2} + V(x) + \mu\varepsilon^\eta H_1\left(x, y, \frac{t}{\varepsilon}\right), \end{aligned} \quad (1)$$

where $\varepsilon > 0$ is a small parameter, $\mu \in \mathbb{R}$, $\eta \geq 0$ and H_0 has a hyperbolic critical point whose invariant manifolds coincide along a separatrix.

This phenomenon was first pointed out by Poincaré [Poi99] but it was not until the last decades when this problem started to be studied rigorously (see for instance [HMS88, SMH91, DS92, Fon93, CG94, Gel94, Fon95, Sau95, DGJS97, DS97, Gel97b, Tre97, GGM99, Gel00, Sau01, DGS04, BF04, BF05, Bal06, Oli06, GOS10, BFGS11]). Nevertheless, the results which obtain asymptotic formulae for the splitting only deal with Hamiltonian systems whose perturbation is an algebraic or trigonometric polynomial with respect to the state variables x and y . In all these cases the splitting of separatrices is exponentially small with respect to the parameter ε . Moreover, the imaginary part of the complex singularity of the time-parametrization of the unperturbed separatrix closest to the real axis plays a significant role.

All the previous works dealing with the periodic case show that under certain non-degeneracy conditions, the distance between the invariant manifolds is of order

$$d \sim \mu\varepsilon^q e^{-\frac{a}{\varepsilon}}, \quad (2)$$

where a is the imaginary part of the complex singularity of the time-parametrization of the unperturbed separatrix closest to the real axis and $q \in \mathbb{R}$. Moreover, for $\eta > \eta^*$, where η^* depends on the properties of both H_0 and H_1 , the splitting is well predicted by the Poincaré–Arnol’d–Melnikov method (see [Mel63, GH83] for a more modern exposition of this method). This case is usually called the *regular case*. In the *singular case* $\eta = \eta^*$ the splitting is exponentially small as (2) but the first order does not coincide with the Melnikov prediction (see [BFGS11] and references therein). In the quasiperiodic case, under certain hypotheses, one can also show that the Melnikov method predicts correctly the splitting and the size of both the Melnikov function and the splitting depends strongly on a [DGJS97, Sau01, DGS04]. However, this case is much less understood and there are very few results.

Nevertheless, many of the models known, for instance in celestial mechanics, are not algebraic or trigonometric polynomials in the state variables but involve functions with a finite strip of analyticity (see, for instance, [LS80, Xia92, MP94, FGKR11]). As far as the authors know, the only result dealing with the exponentially small splitting of separatrices in the periodic case for non-entire perturbations is [Gel97a]. However, the author considers models with a strip of analyticity very big with respect to ε so that he can deal with them as if they were polynomial. In the quasiperiodic case, as far as the authors know, there were no results available up to now.

The goal of this paper is to study how the splitting of separatrices behaviour depends on the width of the analyticity strip when one considers a meromorphic perturbation. Essentially, we see that the size of the splitting depends strongly on this width and that, in general, the singularity of the separatrix does not play any role in this size. We consider also the case when the strip tends to infinity as $\varepsilon \rightarrow 0$ and we see how the size of the splitting tends to the size known for the entire cases. In the other limiting case, namely when the strip of analyticity shrinks to the real line as $\varepsilon \rightarrow 0$, we see that even if the perturbation is still analytic, the splitting becomes algebraic in ε both in the periodic and the quasiperiodic case.

We focus our study in particular examples, which allow us to analyse in great detail the behaviour of the splitting. Nevertheless, we expect the same to happen for fairly general systems.

We work with time periodic and quasiperiodic perturbations of the classical pendulum. More concretely, we consider the following model:

$$\ddot{x} = \sin x + \mu\varepsilon^\eta \frac{\sin x}{(1 + \alpha \sin x)^2} f\left(\frac{t}{\varepsilon}\right), \tag{3}$$

where $f(\tau)$ is an analytic function which depends either periodically or quasiperiodically on τ . Recall that, as stated in (1), $\eta \geq 0$, $\mu \in \mathbb{R}$ and $\varepsilon > 0$.

The associated system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \sin x + \mu\varepsilon^\eta \frac{\sin x}{(1 + \alpha \sin x)^2} f\left(\frac{t}{\varepsilon}\right) \end{aligned} \tag{4}$$

is Hamiltonian with Hamiltonian function

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \cos x - 1 + \mu\varepsilon^\eta \psi(x) f\left(\frac{t}{\varepsilon}\right), \tag{5}$$

where $\psi(x)$ is defined by $\psi'(x) = -\sin x / (1 + \alpha \sin x)^2$ and $\psi(0) = 0$. Here $\alpha \in [0, 1)$ is a parameter which changes the width of the analyticity strip of ψ , which is given by

$$|\operatorname{Im} x| \leq \ln\left(\frac{1 + \sqrt{1 - \alpha^2}}{\alpha}\right). \tag{6}$$

When $\alpha = 0$, the system is entire in x and y and has been previously studied for particular choices of f in [Tre97, DGJS97, OSS03, Oli06], whereas when $\alpha = 1$, ψ is not defined in $x = 3\pi/2$. In this paper, we consider any $\alpha \in (0, 1)$ either independent of or dependent on ε .

In the periodic case, as experts know, the only important property to obtain the asymptotic formula for the splitting is that one has to require that the first harmonics of f are different from zero. Thus, we choose

$$f(\tau) = \sin \tau.$$

Dealing with any other function with non-zero first Fourier coefficients is analogous. In this setting, one can rephrase system (4) as a Hamiltonian system of two degrees of freedom considering $\tau = t/\varepsilon$ as a new angle and I its conjugate action, which gives the Hamiltonian

$$\begin{aligned} K(x, y, \tau, I) &= \frac{I}{\varepsilon} + H(x, y, \tau) \\ &= \frac{I}{\varepsilon} + \frac{y^2}{2} + \cos x - 1 + \mu\varepsilon^\eta \psi(x) \sin \tau. \end{aligned} \tag{7}$$

In the quasiperiodic case, we consider the same model (3) with $f(\tau) = F(\tau, \gamma\tau)$, where

$$\gamma = \frac{\sqrt{5} + 1}{2} \tag{8}$$

is the golden mean number and $F : \mathbb{T}^2 \rightarrow \mathbb{R}$. Note that if one takes $\alpha = 0$, one recovers the model considered in [DGJS97]. On the function F we assume the same hypotheses that are assumed in that paper. Namely, if one considers its Fourier expansion in the angles $\theta = (\theta_1, \theta_2)$,

$$F(\theta_1, \theta_2) = \sum_{k \in \mathbb{Z}^2} F^{[k]} e^{ik \cdot \theta}, \tag{9}$$

we assume that there exist constants $r_1, r_2 > 0$ such that

$$\sup_{k=(k_1, k_2) \in \mathbb{Z}^2} |F^{[k]} e^{r_1|k_1|+r_2|k_2}| < \infty. \quad (10)$$

Furthermore, we assume that there exist a and k_0 such that

$$|F^{[k]}| > a e^{-r_1|k_1|-r_2|k_2|} \quad (11)$$

for all $|k_1|/|k_2|$ which are continuous fraction convergents of γ and $|k_2| > k_0$. An example of a function satisfying these hypotheses is

$$F(\theta_1, \theta_2) = \frac{\cos \theta_1 \cos \theta_2}{(\cosh r_1 - \cos \theta_1)(\cosh r_2 - \cos \theta_2)}.$$

Introducing the angle coordinates (θ_1, θ_2) and their conjugate actions (I_1, I_2) , system (4) can be seen as a three degrees of freedom Hamiltonian system with Hamiltonian

$$\begin{aligned} K(x, y, \theta, I) &= \frac{\omega \cdot I}{\varepsilon} + H(x, y, \theta) \\ &= \frac{\omega \cdot I}{\varepsilon} + \frac{y^2}{2} + \cos x - 1 + \mu \varepsilon^\eta \psi(x) F(\theta_1, \theta_2), \end{aligned} \quad (12)$$

where $\omega = (1, \gamma)$ is the frequency vector.

We have chosen these particular models for several reasons. First, the hyperbolic critical point $(0, 0)$ of the unperturbed pendulum persists when the perturbation is added. This fact is not crucial but simplifies the computations. Second, with the chosen function ψ , the size of the Melnikov function depends on the strip of analyticity of the perturbations, as is expected to happen for general systems. In remark 2.5 in section 2, we consider the non-generic model

$$\ddot{x} = \sin x + \mu \varepsilon^\eta \frac{\sin x}{(1 - \alpha \cos x)^2} \sin \frac{t}{\varepsilon},$$

which has the same strip of analyticity (6). However, due to certain cancellations, the size of Melnikov function does not depend on this strip.

In the quasiperiodic case, we have chosen a very specific function F . On the one hand, we have chosen the frequency vector $\omega = (1, \gamma)$, where γ is the golden mean (8). The size of the splitting strongly depends on the diophantine properties of the chosen frequency vector and, in fact, its rigorous study has been only carried out, as far as the authors know, for quadratic frequency vectors (i.e. frequency vectors such that the ratio between the two frequencies is a quadratic irrational number, see [Sau01, LMS03, DG03]). On the other hand, the chosen function F has finite strip of analyticity in the angles (θ_1, θ_2) . This is the only kind of system for which it is known that the Melnikov function predicts correctly the size of the splitting (see [Sim94, SV01]). In fact, in the quasiperiodic case, the width of the strip of analyticity of F also plays a crucial role in the size of the splitting.

Finally, as we have already explained, this particular choice of the perturbation makes everything easily computable. This allows us to obtain explicit formulae for the first order of the splitting of separatrices, using the Melnikov function, and see how it depends on the width of the analyticity strip of ψ . Then, we can compare our results for α small with the existing previous ones for $\alpha = 0$.

As we have already said, when $\alpha = 1$ system (4) is not defined at $x = 3\pi/2$. Therefore, it makes no sense to study the splitting problem for α too close to 1. Indeed, the perturbation is small in the real line provided

$$\frac{\varepsilon^\eta}{(1 - \alpha)^2} \ll 1.$$

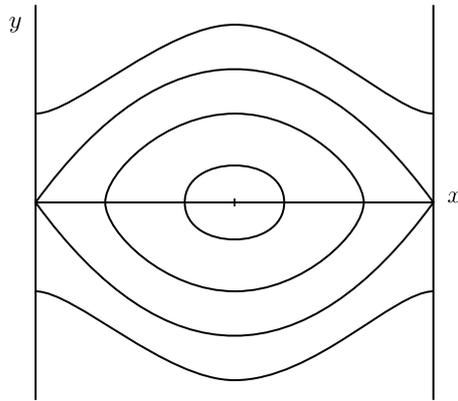


Figure 1. Phase portrait of the pendulum. In it, one can see the two separatrices. We focus our study on the upper one.

Nevertheless, as usually happen in the exponentially small splitting problems (see [GOS10]), we will see that the splitting problem has sense under the slightly weaker hypothesis

$$\frac{\varepsilon^\eta}{(1-\alpha)^{3/2}} \ll 1.$$

When $\mu = 0$, the system is the classical pendulum. It has a hyperbolic critical point at $(0, 0)$ whose invariant manifolds coincide along two separatrices (see figure 1). We focus our attention on the positive one, which can be parametrized as

$$x_0(u) = 4 \arctan(e^u), \quad y_0(u) = \dot{x}_0(u) = \frac{2}{\cosh u}, \quad (13)$$

whose singularities are at $u = i\pi/2 + ik\pi$, $k \in \mathbb{Z}$.

The goal of this paper is to measure how this separatrix splits when one takes $\mu > 0$, paying special attention to how this splitting depends on the relative size between α and ε . The previous results in the periodic case [Tre97, OSS03, Oli06] consider $\alpha = 0$. They obtain an asymptotic formula for the distance between the invariant manifolds, which is of the form

$$d \sim \mu \varepsilon^{\eta-2} e^{-\frac{\pi}{2\varepsilon}}.$$

In this paper, we see that in the periodic case

- If $\alpha \leq \varepsilon^2$, that is when the strip of analyticity is of order $2 \ln(1/\varepsilon)$ or bigger, the distance between the invariant manifolds coincides with the results obtained in the case $\alpha = 0$, when the perturbation is a trigonometric polynomial.
- If $\varepsilon^2 \ll \alpha$ and $1 - \alpha \gg \varepsilon^2$, that is when the strip of analyticity is between $2 \ln(1/\varepsilon)$ and ε , the distance is exponentially small but the first order is given by a completely different formula

$$d \sim \mu \varepsilon^q e^{-\frac{c}{\varepsilon}},$$

where $q \in \mathbb{R}$ and $0 < c < \pi/2$ are constants which depend on α . Therefore, the splitting is bigger than it was in the polynomial case. Moreover, we see that it monotonically increases with α .

- Finally, if $0 < 1 - \alpha \leq \varepsilon^2$, that is when the strip of analyticity is narrower than ε , the distance between the invariant manifolds is non-exponentially small,

$$d \sim \mu \varepsilon^q,$$

where $q \in \mathbb{R}$ is a constant which depends on α .

In particular, the second statement shows that even if α is a small parameter, our study prevents the use of the classical approach in perturbation theory. It consists in expanding the perturbation term in powers of α and studying the splitting of the first order, which has a polynomial perturbation. We will see that even for $\alpha \sim \varepsilon$, this approach leads to a wrong result for the splitting. Therefore, we show that the study of the splitting of separatrices for meromorphic perturbations cannot be reduced to the study of simplified polynomial models. In the quasiperiodic case, an analogous phenomenon happens.

The structure of the paper goes as follows. First in section 2 we give the main results considering the periodic case. In proposition 2.1 we show the behaviour of the Melnikov function with respect to α . In corollary 2.6 we analyse the range of the parameter α for which the Melnikov function is not exponentially small. Then, in theorems 2.8 and 2.10, we show for which range of parameters ε , α and η , the Melnikov function correctly predicts the splitting.

In section 3 we consider the quasiperiodic case. First in proposition 3.1 we study the size of the Melnikov function and in theorem 3.6 we prove its validity. As in the periodic case, we also show that if $1 - \alpha \geq \varepsilon^2$ both the Melnikov function and the splitting of separatrices are not exponentially small. These results are given in corollary 3.4 and theorem 3.8, respectively.

Section 4 is devoted to prove theorems 2.8 and 2.10, and section 5 is devoted to prove theorems 3.6 and 3.8.

Finally, in the appendix we give some heuristic ideas about how to deal with the so-called singular periodic case, namely, when the parameter η reaches a certain limiting value and therefore the Melnikov function does not predict correctly the size of the splitting.

1.1. Heuristics on the relation between regularity and Arnol'd diffusion

Even if in this paper we study the so-called isochronous case, where the frequency of the perturbation is fixed, the same kind of study would apply to an anisochronous case (see [Sau01]). In the anisochronous case one can encounter both situations, the case of rationally dependent frequencies, which leads to a periodic in time perturbation, and the case of rationally independent frequencies, which leads to a quasiperiodic perturbation.

Anisochronous systems have attracted a lot of attention because they are good models to study the phenomenon called Arnol'd diffusion. The name comes from the fact that it was Arnol'd who produced in 1964 [Arn64] the first example showing a possible mechanism that leads to global instabilities in nearly integrable Hamiltonian systems.

Arnol'd proved the presence of instabilities in the following particular model:

$$H(I_1, I_2, \varphi_1, \varphi_2, t) = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon(\cos \varphi_1 - 1) + \mu\varepsilon(\cos \varphi_1 - 1)(\sin \varphi_2 + \sin t),$$

showing the existence of orbits whose action I_2 changes drastically.

Nevertheless, it is expected that instabilities exist in fairly general nearly integrable Hamiltonian systems. Chierchia and Gallavotti in [CG94] proposed the study of the following generalization of the Arnol'd model:

$$H(I_1, I_2, \varphi_1, \varphi_2, t) = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon(\cos \varphi_1 - 1) + \mu\varepsilon h(\varphi_1, \varphi_2, t; \varepsilon). \quad (14)$$

In this setting, they coined the terminology *a priori stable* versus *a priori unstable* systems. *A priori stable* systems are those of the form (14) with the parameter μ satisfying $\mu = \varepsilon^\eta$ with $\eta \geq 0$ and *a priori unstable* are those with parameters $\varepsilon = 1$ and μ small. In the *a priori stable* case the unperturbed system, $\varepsilon = 0$, is completely integrable in the sense that it is written in global action-angle variables. In the *a priori unstable* case the unperturbed system, $\mu = 0$,

even if it is integrable in the sense that it has conserved quantities, presents some hyperbolicity, namely has partially hyperbolic tori with homoclinic trajectories.

There have been some recent works proving the existence of instabilities for *a priori* unstable systems using the ideas proposed by Arnol'd (see [DdlLS06, DH09] and see [CY04, Tre04, Ber08] for proofs using other methods). One of the main steps in the proof is to see that the stable and unstable manifolds of the partially hyperbolic tori, which coincide when $\mu = 0$, split producing chains of heteroclinic orbits. To detect the splitting of these invariant manifolds becomes an essential step in these geometric methods. The main reason that makes it very difficult to detect this splitting when μ and ε are small is that one expects this splitting to be exponentially small in ε and, thus, very difficult to study. This was the origin of the distinction between *a priori* stable and unstable systems, Arnol'd overcame this difficulty taking the parameter μ exponentially small in ε . It is an open problem to see whether the Arnol'd mechanism works in the *a priori* stable setting, namely taking $\mu = \varepsilon^\eta$ with $\eta \geq 0$.

Nonetheless, the regularity of the system plays a crucial role in the classification between *a priori* stable and unstable systems. The first observation, that is commonly accepted, is that in the C^r case, even in the *a priori* stable setting, the splitting is not exponentially small. Nevertheless, as far as the authors know, the only proof of this fact is the paper [DJS99] where it is seen that for quasiperiodic C^r perturbations the splitting is polynomially small in ε . Therefore the distinction between *a priori* stable and unstable systems regarding the splitting problem only has sense for analytic perturbations. Moreover, the results stated in theorems 2.10 and 3.8 show that for analytic perturbations with narrow strip of analyticity, the size of the splitting is not exponentially small. In particular, we see that the splitting is polynomial in ε in both the periodic and the quasiperiodic case. Therefore, one would expect that for anisochronous systems (14) with a narrow strip of analyticity the splitting at the resonances and in the non-resonant zones are of the same order and polynomial with respect to ε .

On the other hand, it is a commonly accepted fact that the bigger the size of the splitting the faster the diffusion. In fact, Nekhoroshev type lower bounds of the diffusion time (or upper bounds of the stability time) increase with the regularity of the system [Nek77, MS02, Bou10] and this agrees with the fact that the splitting decreases with the regularity. For analytic Hamiltonians, these bounds are bigger for bigger strips of analyticity [Pös93, DG96]. The results in this paper show that this is also consistent with the fact that the splitting decreases when the strip of analyticity increases.

2. The Melnikov function and its validity in the periodic case

As we want to deal with a time periodic perturbation of the pendulum equation, the dynamics is better understood in the three-dimensional extended phase space $(x, y, \tau) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$. In this space, $\Lambda = \{(0, 0, \tau); \tau \in \mathbb{T}\}$ is a hyperbolic periodic orbit and the unperturbed upper separatrix given in (13) becomes in this setting a two-dimensional manifold, which is the intersection of the upper branches of the two-dimensional stable and unstable invariant manifolds. They can be parametrized as

$$\begin{aligned} \mathcal{W}^{u,+}(\Lambda) &= \mathcal{W}^{s,+}(\Lambda) = \{(x, y, \tau) : H_0(x, y) = 0, y > 0\} \\ &= \{(x, y, \tau) = (x_0(u), y_0(u), \tau); (u, \tau) \in \mathbb{R} \times \mathbb{T}\}, \end{aligned}$$

where $(x_0(u), y_0(u))$ is the parametrization of the separatrix given in (13).

Our goal is to study how these manifolds split when $\mu \neq 0$. To this end we need to introduce some notion of distance between them. As the manifolds are graphs for $\mu = 0$, the same happens for $\mu\varepsilon^\eta > 0$ small enough in suitable domains. More concretely, in section 4

we will see that one can parametrize the perturbed stable and unstable manifolds as

$$\begin{aligned}x &= x_0(u) \\ y &= y^{u,s}(u, \tau) \\ \tau &= \tau.\end{aligned}$$

Here, $(u, \tau) \in (-\infty, U) \times \mathbb{T}$, for certain $U > 0$, for the unstable manifold and $(u, \tau) \in (-U, \infty) \times \mathbb{T}$ for the stable one. Taking into account that the invariant manifolds are Lagrangian, in section 4 we use the Hamilton–Jacobi equation to see that the functions $y^{u,s}$ can be given as

$$y^{u,s}(u, \tau) = \frac{1}{y_0(u)} \partial_u T^{u,s}(u, \tau),$$

for certain generating functions $T^{u,s}$.

Therefore, a natural way to measure the difference between the manifolds is to compute

$$D(u, \tau) = \partial_u T^s(u, \tau) - \partial_u T^u(u, \tau).$$

If one considers a perturbative approach taking μ as small parameter, one can easily see that the first order in μ of the function $D(u, \tau)$ is given by the Melnikov function

$$\mathcal{M}(u, \tau) = \int_{-\infty}^{+\infty} \{H_0, H_1\} \left(x_0(u+s), y_0(u+s), \tau + \frac{s}{\varepsilon} \right) ds, \quad (15)$$

where H_0 and H_1 are the Hamiltonians given in (1).

In other words, one has that

$$D(u, \tau) = \mu \varepsilon^\eta \mathcal{M}(u, \tau) + \mathcal{O}(\mu^2 \varepsilon^{2\eta}). \quad (16)$$

To see that the manifolds split, we can choose a transversal section to the unperturbed separatrix to measure their distance. The simplest one is $x = \pi$, which corresponds to computing $D(0, \tau)$ (see (13)). Then, the zeros of $D(0, \tau)$ correspond to homoclinic orbits of the perturbed system and the distance between the invariant manifolds is given by

$$d(\tau) = y^s(0, \tau) - y^u(0, \tau) = \frac{1}{2} D(0, \tau). \quad (17)$$

Nevertheless, when α is close to 1, namely $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, the perturbative term in (3) has a non-uniform bound for $x \in [0, 2\pi]$. Indeed, its maximum (in absolute value), which is $\mu\varepsilon^\eta / (1 - \alpha)^2 = \mu\varepsilon^{\eta-2r} / C^2$, is reached at $x = 3\pi/2$. Therefore, it is natural to expect that the invariant manifolds of the perturbed system remain $\mu\varepsilon^\eta$ -close to the unperturbed separatrix only before they reach a neighbourhood of the section $x = 3\pi/2$. For this reason, in this case we will measure the distance in this section, where one expects that the perturbed manifolds are closer. This section, by (13), corresponds to $u = \ln(1 + \sqrt{2})$. Therefore, we define $\tilde{d}(\tau)$, the distance between the perturbed invariant manifolds at the section $x = 3\pi/2$, which is given by

$$\tilde{d}(\tau) = y^s(\ln(1 + \sqrt{2}), \tau) - y^u(\ln(1 + \sqrt{2}), \tau) = 2\sqrt{2} D(\ln(1 + \sqrt{2}), \tau). \quad (18)$$

For α close to 1, one would expect that the distance $d(\tau)$ is bigger than $\tilde{d}(\tau)$ since the manifolds deviate from the homoclinic after they cross the section $x = 3\pi/2$. Nevertheless, when α is not close to 1, both quantities are equivalent.

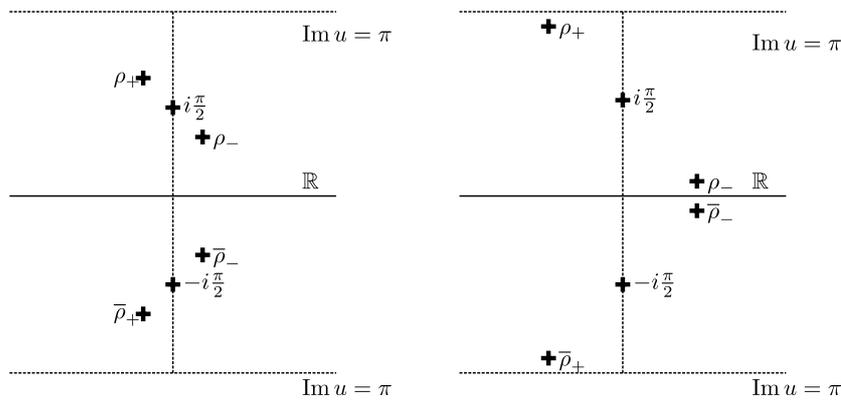


Figure 2. Location of the singularities $u = \rho_-, \rho_+, \bar{\rho}_-, \bar{\rho}_-$. In the left picture we show them for α small, in such a way that they are close to the singularities of the separatrix of the pendulum $u = \pm i\pi/2$. In the right picture we show them for α close to 1. Then, ρ_- and $\bar{\rho}_-$ approach the real axis while ρ_+ and $\bar{\rho}_+$ approach the lines $\text{Im } u = \pi$ and $\text{Im } u = -\pi$, respectively.

2.1. The Melnikov function

In this section we compute the Melnikov function associated with Hamiltonian (5) with $f(\tau) = \sin \tau$ and we study its dependence on ε and α . We recall that the Melnikov function (15) is given by

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = 4 \int_{-\infty}^{+\infty} \frac{\sinh(u+s) \cosh(u+s)}{(\cosh^2(u+s) - 2\alpha \sinh(u+s))^2} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds. \tag{19}$$

As is well known, the evaluation of this integral can be performed using residuum theory. To this end, one has to look for the poles of the function

$$\beta(u) = \frac{\sinh(u) \cosh(u)}{(\cosh^2(u) - 2\alpha \sinh(u))^2} \tag{20}$$

closest to the real axis. For $\alpha = 0$, $\beta(u)$ has order 3 poles at $\rho_{\pm}^0 = \pm i\pi/2$. Nevertheless, for $\alpha > 0$ these poles bifurcate into a combination of zeros and poles. Since the size of the Melnikov function (19) depends on the location of these poles, one has to study their dependence on α . We will see that the relative size between the parameter α and the period $2\pi\varepsilon$ of the perturbation will lead to significantly different sizes of (19).

Considering the denominator of $\beta(u)$, namely $(\cosh^2(u) - 2\alpha \sinh(u))^2$, one can easily see that the poles of (20) are the solutions of

$$\sinh u = \alpha \pm i\sqrt{1 - \alpha^2},$$

This equation has four families of solutions given by $\rho_- + 2\pi ki, \rho_+ + 2\pi ki$, which are solutions of

$$\sinh u = \alpha + i\sqrt{1 - \alpha^2},$$

and their conjugate families $\bar{\rho}_- + 2\pi ki$ and $\bar{\rho}_+ + 2\pi ki$ for $k \in \mathbb{Z}$. The singularities $\rho_-, \rho_+, \bar{\rho}_-$ and $\bar{\rho}_+$ are the closest to the real axis (see figure 2) and they satisfy

$$0 < \text{Im } \rho_- < \frac{\pi}{2} < \text{Im } \rho_+ < \pi,$$

for $\alpha \in (0, 1)$.

In particular, if one considers α as a small parameter, they have the expansions

$$\rho_{\pm}(\alpha) = i\frac{\pi}{2} \pm (-1 + i)\sqrt{\alpha} + \mathcal{O}(\alpha) \quad \text{as } \alpha \rightarrow 0. \tag{21}$$

In the other limiting regime $\alpha \rightarrow 1$, their expansions are

$$\begin{aligned} \rho_{-}(\alpha) &= \ln(1 + \sqrt{2}) + i(1 - \alpha)^{1/2} + \mathcal{O}(1 - \alpha) \\ \rho_{+}(\alpha) &= -\ln(1 + \sqrt{2}) + \pi i - i(1 - \alpha)^{1/2} + \mathcal{O}(1 - \alpha) \end{aligned} \quad \text{as } \alpha \rightarrow 1. \tag{22}$$

Then, one can see that the function β for $-\pi < |\text{Im } u| < \pi$ and $-1 < \text{Re } u < 1$ behaves as

$$\beta(u) \sim \frac{(u - i\pi/2)(u + i\pi/2)}{(u - \rho_{-})^2(u - \rho_{+})^2(u - \bar{\rho}_{-})^2(u - \bar{\rho}_{+})^2}. \tag{23}$$

Using this fact, one can easily compute the size of the Melnikov function in (19) using residuum theory.

Proposition 2.1. *There exists $\varepsilon_0 > 0$ such that $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$,*

- *If α satisfies $0 < \alpha \leq C\varepsilon^v$ where $v > 2$ and $C > 0$ are constants independent of ε , the Melnikov function (19) satisfies the asymptotic formula*

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \frac{4\pi}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \left(\cos(\tau - u/\varepsilon) + \mathcal{O}\left(\frac{\alpha}{\varepsilon^2}, e^{-\frac{\pi}{2\varepsilon}}\right) \right). \tag{24}$$

- *If $\alpha = \alpha_*\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ for some constant $\alpha_* > 0$, the Melnikov function (19) satisfies the asymptotic formula*

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \frac{\lambda}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \sin(\tau - \phi_* - u/\varepsilon) + \mathcal{O}\left(\frac{1}{\varepsilon} e^{-\frac{\pi}{2\varepsilon}}\right) \tag{25}$$

for certain constants $\lambda = \lambda(\alpha_*)$ and $\phi_* = \phi_*(\varepsilon, \alpha_*)$.

- *If α satisfies $C\varepsilon^v < \alpha < 1$ where $v \in (0, 2)$ and $C > 0$ are constants independent of ε , the Melnikov function (19) satisfies the asymptotic formula*

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \left| \frac{\delta_2(\alpha)}{\varepsilon} + \delta_1(\alpha) \right| e^{-\frac{\text{Im } \rho_{-}}{\varepsilon}} \sin(\tau - \phi - u/\varepsilon) + \mathcal{O}\left(e^{-\frac{\pi}{2\varepsilon}}\right). \tag{26}$$

where $\delta_i(\alpha)$ are given by

$$\delta_1(\alpha) = \frac{2\pi}{(1 - \alpha^2)^{3/2}} \left(\sinh \rho_{-} - i(1 - \alpha^2)^{1/2} \right) \tag{27}$$

$$\delta_2(\alpha) = \frac{2\pi \sinh \rho_{-}}{(1 - \alpha^2) \cosh \rho_{-}} \tag{28}$$

with $\sinh \rho_{-} = \alpha + i\sqrt{1 - \alpha^2}$, and $\phi = \phi(\varepsilon, \alpha)$.

When $\alpha \ll \varepsilon^2$, the first statement of proposition 2.1 ensures that the Melnikov function is non-degenerate. In the case $\alpha \sim \varepsilon^2$, one would have to analyse the behaviour of the constant $\lambda(\alpha_*)$, which appears in formula (25), to check that it does not vanish. Even if this constant is computable in our example, we will not study it, since it is not the purpose of this paper to study this particular case. We just want to note that for general α_* , the first asymptotic order of the Melnikov function has changed from (24) to (25). In fact, as can be seen in formula (26), $\alpha \sim \varepsilon^2$ is the transition value at which the size of the Melnikov function changes drastically. Indeed, formula (26) and the expansion of ρ_{-} given in (22) show that the size of the Melnikov function becomes bigger when α approaches 1. This particular behaviour will be analysed in section 2.1.1.

The next corollary, whose proof is straightforward, analyses the behaviour of formula (26) for α in the range $\varepsilon^2 \ll \alpha \leq \alpha_0 < 1$ for any fixed $\alpha_0 \in (0, 1)$. We observe that ρ_{-} has the asymptotic expansion (21) when $\alpha \rightarrow 0$. The next corollary shows that the Melnikov function is exponentially small for this range of parameter α .

Corollary 2.2. *We fix any $\alpha_0 \in (0, 1)$. Then, the Melnikov function in (26) has the following asymptotic formulae for $\varepsilon^2 \ll \alpha \leq \alpha_0 < 1$.*

- *If α satisfies $\alpha = C\varepsilon^\nu$ for certain $\nu \in (0, 2)$ and $C > 0$,*

$$\mathcal{M}(u, \tau; \varepsilon, C\varepsilon^\nu) = \frac{|\delta_2^0|}{C\varepsilon^{1+\nu/2}} e^{-\frac{\text{Im } \rho_-}{\varepsilon}} (\sin(\tau - \phi - u/\varepsilon) + \mathcal{O}(\varepsilon^{\nu/2})),$$

where

$$\delta_2^0 = -\pi(1 + i).$$

Therefore, in this case the Melnikov function is non-degenerate since $\delta_2^0 \neq 0$.

- *If $\alpha \in (0, \alpha_0]$ is independent of ε ,*

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \frac{|\delta_2(\alpha)|}{\varepsilon} e^{-\frac{\text{Im } \rho_-}{\varepsilon}} (\sin(\tau - \phi - u/\varepsilon) + \mathcal{O}(\varepsilon)).$$

It can be checked that $\delta_2(\alpha) \neq 0$ for any $\alpha \in (0, 1)$, and therefore, in this case the Melnikov function is also non-degenerate.

Remark 2.3. If one takes $\alpha \sim \varepsilon^\nu$ and $\nu \rightarrow 2$ in formula (26), one does not obtain formula (25). The reason is that formula (26) only takes into account the residuum of ρ_- since the residuum of ρ_+ is exponentially small with respect to the one of ρ_- . Nevertheless, this is not the case when $\alpha \sim \varepsilon^2$ (see (21)). Then, the residua of both singularities ρ_- and ρ_+ make a contribution to the Melnikov function of the same exponentially small order $\mathcal{O}(\varepsilon^{-2}e^{-\pi/2\varepsilon})$.

Remark 2.4. In the first statement of proposition 2.1, one can see that, when the strip of analyticity (6) is *wide enough*, namely taking $\alpha \sim \varepsilon^\nu$ with $\nu > 2$, the Melnikov function at first order behaves as in the entire case $\alpha = 0$ [Gel97a]. In other words, the Melnikov function is not sensitive to the finiteness of the strip of analyticity and the exponentially small coefficient is given by the imaginary part of the singularities of the separatrix.

On the other hand, when the strip of analyticity (6) is independent of ε or not extremely big with respect to ε , namely taking $\alpha \sim \varepsilon^\nu$ with $\nu \in [0, 2)$, the exponentially small coefficient does not coincide with the imaginary part of the singularity of the unperturbed separatrix. Instead it is given by the imaginary part of this new singularity ρ_- , which appears when one evaluates the perturbation along the unperturbed separatrix. Note that even if one takes, for instance, $\alpha = \varepsilon$, the strip of analyticity is of order $\ln(1/\varepsilon)$ and one has that

$$\mathcal{M}(u, \tau; \varepsilon, \varepsilon) \sim \varepsilon^{-\frac{3}{2}} e^{-\frac{\pi-2\sqrt{\varepsilon}}{2\varepsilon}}.$$

That is, even for perturbations with a wide strip of analyticity with respect to ε , a correcting term appears in the exponential.

The case $\alpha \sim \varepsilon^2$ is the boundary between these two different types of behaviour. In this case, the exponential coefficient is given by the imaginary part of the singularity of the unperturbed separatrix but the constant in front of the exponential is not the one given in the first statement of proposition 2.1 but a different one, which is given in the second statement for the value $\alpha = \alpha_*\varepsilon^2 + \mathcal{O}(\varepsilon^3)$.

Proof of proposition 2.1. The classical way to compute the Melnikov function is to change the path of integration to $\text{Im } s = \pm\pi$ and apply residuum theory, using that the function β has order two poles at $\rho_-, \rho_+, \bar{\rho}_-$ and $\bar{\rho}_+$ in the strip $-\pi < \text{Im } s < \pi$. Nevertheless, to prove the first statement, instead of computing the residua of both ρ_\pm and $\bar{\rho}_\pm$ and looking for the

cancellations, we just expand the Melnikov integral in power series of α , since it is uniformly convergent as a real integral. Thus, we obtain,

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = 4 \sum_{n=0}^{\infty} (n+1)2^n \alpha^n \int_{-\infty}^{+\infty} \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+3}(u+s)} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds.$$

Then, the term for $n = 0$ gives the first asymptotic order and can be computed using residuum theory. To bound the other terms goes as follows. We Fourier-expand the terms in the series of the Melnikov integral in τ and we change the path of integration to $\text{Im } s = \pi/2 - \varepsilon$ or $\text{Im } s = -(\pi/2 - \varepsilon)$ depending on the harmonic. This gives us the exponentially small term $e^{-\frac{\pi}{2\varepsilon}}$. To bound the other terms in the integral, we use that

$$\left| \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+1}(u+s)} \right| \leq \left(\frac{K}{\varepsilon}\right)^{2n+1},$$

for certain constant $K > 0$ independent of ε . Moreover, using that

$$\frac{1}{|\cosh^2(u+s)|}$$

decays exponentially as $\text{Re } s \rightarrow \pm\infty$ and that it has poles of order two $u+s = \pm i\pi/2$, one can see that

$$\int_{-\infty}^{\infty} \frac{1}{|\cosh^2(u+s \pm (i\pi/2 - \varepsilon))|} ds \leq \frac{K}{\varepsilon}.$$

Then, one has that

$$\left| \int_{-\infty}^{+\infty} \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+3}(u+s)} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds \right| \leq \left(\frac{K}{\varepsilon}\right)^{2n+2} e^{-\frac{\pi}{2\varepsilon}}.$$

Therefore, the remainder can be easily bounded provided $0 < \alpha \ll \varepsilon^2$.

When $\alpha \sim \varepsilon^2$ we cannot ensure that the preceding series is convergent and therefore we use residuum theory directly in the strip $0 < \text{Im } s < \pi$ or $-\pi < \text{Im } s < 0$ depending on the harmonic. For instance, for the positive harmonic, we take the strip $0 < \text{Im } s < \pi$, which contains the singularities ρ_- and ρ_+ , that are at a distance of order $\mathcal{O}(\varepsilon)$ from $i\pi/2$ (see (21)). Then, it is enough to compute the residuum of $\beta(s)e^{is/\varepsilon}$ at these singularities, which using (23) satisfies

$$\text{Res}(\beta(s)e^{is/\varepsilon}, s = \rho_{\pm}) = \frac{A_{\pm}}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon} e^{-\frac{\pi}{2\varepsilon}}\right).$$

for certain computable constants $A_{\pm} \in \mathbb{C}$ independent of ε .

In the case $\alpha \sim \varepsilon^{\nu}$ with $0 \leq \nu < 2$, we have that $\pi/2 - \text{Im } \rho_-, \text{Im } \rho_+ - \pi/2 \sim \varepsilon^{\nu/2} \gg \varepsilon$. Therefore,

$$e^{-\frac{\text{Im } \rho_+}{\varepsilon}} \ll e^{-\frac{\pi}{2\varepsilon}} \ll e^{-\frac{\text{Im } \rho_-}{\varepsilon}}.$$

Using these facts, we can compute the Melnikov integral changing the path up to $\text{Im } s = \pi/2$ and we just need to consider the residuum at the singularity ρ_- , which can be explicitly computed. □

Remark 2.5. All the singularities of the function $\beta(u)$ in (20) have different imaginary parts for any $\alpha \in (0, 1)$. This is one of the reasons for the choice of the perturbation in (3), since this is not always the case. We consider, for instance, the pendulum with a different perturbation as

$$\ddot{x} = \sin x + \mu\varepsilon^{\eta} \frac{\sin x}{(1 - \alpha \cos x)^2} \sin \frac{t}{\varepsilon}, \tag{29}$$

which has Hamiltonian function

$$\tilde{H}\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \cos x - 1 + \mu\varepsilon^\eta \frac{1}{\alpha(1 - \alpha \cos x)} \sin \frac{t}{\varepsilon}.$$

Then, the Melnikov function is given by

$$\tilde{\mathcal{M}}\left(u, \frac{t}{\varepsilon}\right) = 4 \int_{-\infty}^{+\infty} \frac{\cosh(u+s) \sinh(u+s)}{\left((1-\alpha) \cosh^2(u+s) + 2\alpha\right)^2} \sin\left(\frac{t+s}{\varepsilon}\right) ds$$

To compute its size, one has to study the singularities of

$$\tilde{\beta}(u) = \frac{\cosh u \sinh u}{\left((1-\alpha) \cosh^2 u + 2\alpha\right)^2}.$$

The closest ones to the real axis with positive imaginary part are

$$\tilde{\rho}_\pm = i\frac{\pi}{2} \pm \operatorname{arcsinh} \sqrt{\frac{2\alpha}{1-\alpha}}$$

and therefore both have the same imaginary part for any $\alpha \in (0, 1)$. In these cases, the Melnikov function, which is given by the residua of both singularities, has the same size for any α satisfying $0 \leq \alpha \leq \alpha_0 < 1$ for any α_0 independent of ε , but is divergent for $\alpha = 1$.

2.1.1. Narrow strip of analyticity: a drastic change in the size of the Melnikov function. In corollary 2.2 we have seen that the Melnikov function is exponentially small with respect to ε provided $0 < \alpha \leq \alpha_0 < 1$ for any fixed $\alpha_0 \in (0, 1)$. We devote this section to study this function when α is close to 1, which will lead to a Melnikov function which is either exponentially small with a different exponential dependence on ε or even to a Melnikov function that is not only exponentially small but also tends to infinity as $\varepsilon \rightarrow 0$.

We consider $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$. In this setting, one can see that the analyticity strip (6) of the Hamiltonian system (4) is very small, of order $\mathcal{O}(\varepsilon^{r/2})$. As a consequence, the singularity $\rho_-(\alpha)$ is very close to the real line. Indeed, from (22), one has

$$\operatorname{Im} \rho_-(\alpha) = C^{1/2} \varepsilon^{r/2} + \mathcal{O}(\varepsilon^r).$$

Corollary 2.6. *If $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, the Melnikov function in (26) has the following asymptotic formulae:*

- If $0 < r < 2$

$$\mathcal{M}(u, \tau; \varepsilon, 1 - C\varepsilon^r) = \frac{\pi}{\sqrt{2}C\varepsilon^{1+r}} e^{-\frac{\operatorname{Im} \rho_-}{\varepsilon}} \left(\sin(\tau - \phi - u/\varepsilon) + \mathcal{O}(\varepsilon^{r/2}, \varepsilon^{1-r/2}) \right).$$

- If $r = 2$,

$$\mathcal{M}(u, \tau; \varepsilon, 1 - C\varepsilon^2) = \frac{\pi e^{-\sqrt{C}}}{\sqrt{2}C^{3/2}\varepsilon^3} \left(\sin(\tau - \phi - u/\varepsilon) + \mathcal{O}(\varepsilon) \right),$$

- If $r > 2$,

$$\mathcal{M}(u, \tau; \varepsilon, 1 - C\varepsilon^r) = \frac{\pi}{\sqrt{2}C^{3/2}\varepsilon^{3r/2}} \left(\sin(\tau - \phi - u/\varepsilon) + \mathcal{O}(\varepsilon^{r/2-1}) \right),$$

where $\phi = \phi(\varepsilon, \alpha)$ is the constant given in proposition 2.1.

Therefore, in all these cases the Melnikov function is non-degenerate.

Note that in the last two cases, the Melnikov function is not small but goes to infinity as $\varepsilon \rightarrow 0$. Nevertheless, this does not imply that the distance between the perturbed invariant manifolds blows up. The reason is that, as will be seen in section 2.2.1, $\mu\varepsilon^\eta \mathcal{M}$ gives the first order of this distance provided $\eta > 3r/2$ and, under these assumptions, $\mu\varepsilon^\eta \mathcal{M}$ is small.

Remark 2.7. To illustrate which size the Melnikov function has for the range of α considered in the first statement of corollary 2.6, we can take $\alpha = 1 - \varepsilon$. Then, using expansion (22), one can see that

$$\mathcal{M}(u, \tau; \varepsilon, 1 - \varepsilon) \sim \varepsilon^{-2} e^{-\frac{1}{\sqrt{\varepsilon}}}.$$

Namely, the Melnikov function is still exponentially small but it has a different exponential dependence on ε .

2.2. Validity of the Melnikov function

Once we have computed the Melnikov function in proposition 2.1, provided $0 < \alpha \leq \alpha_0 < 1$ for a fixed α_0 , we can compute the prediction it gives for the distance between the manifolds at the section $x = \pi$, which we call $d_0(\tau)$. Recall that by (16) it is the first order in μ of the function $d(\tau)$ given in (17).

If α satisfies $0 < \alpha \leq C\varepsilon^\nu$ where $\nu > 2$ and $C > 0$ are constants independent of ε , $d_0(\tau)$ is given by

$$d_0(\tau) = 2\pi\mu\varepsilon^{\eta-2} e^{-\frac{\pi}{2\varepsilon}} \left(\cos \tau + \mathcal{O}\left(\frac{\alpha}{\varepsilon^2}, e^{-\frac{\pi}{2\varepsilon}}\right) \right).$$

If $\alpha = \alpha_*\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ for any constant $\alpha_* > 0$, $d_0(\tau)$ is given by

$$d_0(\tau) = \frac{\lambda(\alpha_*)}{2} \mu\varepsilon^{\eta-2} e^{-\frac{\pi}{2\varepsilon}} \sin(\tau - \phi_*) + \mathcal{O}\left(\mu\varepsilon^{\eta-1} e^{-\frac{\pi}{2\varepsilon}}\right).$$

Finally, if α satisfies $C\varepsilon^\nu < \alpha \leq \alpha_0 < 1$ where $\nu \in (0, 2)$, $C > 0$ and $\alpha_0 \in (0, 1)$ are constants independent of ε , $d_0(\tau)$ is given by

$$d_0(\tau) = \frac{|\delta_2(\alpha)|}{2} \mu\varepsilon^{\eta-1} e^{-\frac{\text{Im}\rho}{\varepsilon}} \sin(\tau - \phi) + \mathcal{O}\left(\mu\varepsilon^\eta e^{-\frac{\text{Im}\rho}{\varepsilon}}\right).$$

As we have already explained, direct application of Melnikov theory only ensures that the distance between the manifolds is given by

$$d(\tau) = d_0(\tau) + \mathcal{O}\left(\mu^2\varepsilon^{2\eta}\right).$$

Therefore, the Melnikov function is the first order of the splitting provided μ is exponentially small with respect to ε . However, it is well known that often, even if Melnikov theory cannot be applied, the Melnikov function is the true first order of the splitting (see [HMS88, DS92, Gel94, DS97, Gel97a, BF04, BF05, Gel00, GOS10, BFGS11]). The next theorem shows that, under certain conditions, the Melnikov function gives the true first order of the distance between the manifolds. We want to point out that the only available proof of the correct prediction of the Melnikov function for meromorphic perturbations is [Gel97a], in which the author considers systems with analyticity strip wide enough with respect to ε . All the other references deal with polynomial perturbations. Thus, this theorem is, as far as the authors know, the first one which shows the dependence of the size of the splitting on the width of the analyticity strip.

Theorem 2.8. *We consider any $\mu_0 > 0$ and $\alpha_0 \in (0, 1)$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$, $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, \alpha_0]$ such that $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ is small enough,*

- *If α satisfies $0 < \alpha \leq C\varepsilon^\nu$ where $\nu > 2$ and $C > 0$ are constants independent of ε , the invariant manifolds split and their distance at the section $x = \pi$ is given by*

$$d(\tau) = 2\pi\mu\varepsilon^{\eta-2} e^{-\frac{\pi}{2\varepsilon}} \left(\cos \tau + \mathcal{O}\left(\frac{\alpha}{\varepsilon^2}, \mu\varepsilon^\eta\right) \right).$$

- If $\alpha = \alpha_* \varepsilon^2 + \mathcal{O}(\varepsilon^3)$ for some constant $\alpha_* > 0$ and the constant $\lambda(\alpha_*)$ introduced in proposition 2.1 satisfies $\lambda(\alpha_*) \neq 0$, the invariant manifolds split and their distance at the section $x = \pi$ is given by

$$d(\tau) = \frac{|\lambda(\alpha_*)|}{2} \mu \varepsilon^{\eta-2} e^{-\frac{\pi}{2\varepsilon}} (\sin(\tau - \phi_*) + \mathcal{O}(\mu \varepsilon^\eta)).$$

- If α satisfies $C \varepsilon^\nu < \alpha < \alpha_0$ where $\nu \in (0, 2]$ and $C > 0$ are constants independent of ε , the invariant manifolds split and their distance at the section $x = \pi$ is given by

$$d(\tau) = \frac{|\delta_2(\alpha)|}{2} \mu \varepsilon^{\eta-1} e^{-\frac{\text{Im} \rho}{\varepsilon}} (\sin(\tau - \phi) + \mathcal{O}(\mu \varepsilon^{\eta-1} \sqrt{\alpha}, \varepsilon)).$$

The proof of this theorem is deferred to section 4.

Remark 2.9. If one considers $\alpha \sim \varepsilon^\nu$ with $\nu > 2$ the condition $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ small enough is equivalent to requiring $\eta > 0$. This condition is the same as has to be required if one considers the polynomial case $\alpha = 0$ (see [Tre97, BFGS11]). In other words, in the case $\nu > 2$ the validity of the Melnikov function is the same as in the polynomial case.

On the other hand, if one considers $\alpha \sim \varepsilon^\nu$ with $\nu < 2$, the condition $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ small enough is

$$\eta - 1 + \frac{\nu}{2} > 0.$$

In particular, if one considers α as a parameter independent of ε , one has to require $\eta > 1$.

The limit cases in the previous settings, $\eta = 0$ and $\eta - 1 + \nu/2 = 0$, respectively, are usually called singular cases (see [BFGS11]). In the appendix we make some remarks about how these cases could be studied.

2.2.1. Narrow strip of analyticity: validity of the Melnikov function. In section 2.1.1 we have seen how the Melnikov function changes its size drastically when $\alpha = 1 - C\varepsilon^r$ with $r > 0$, even becoming unbounded as $\varepsilon \rightarrow 0$ if $r \geq 2$. In this section we prove that $\mu \varepsilon^\eta \mathcal{M}$ gives the correct first order for the splitting of separatrices. Note that, in the range of parameters for which the Melnikov function is big, since $\mu \varepsilon^\eta \mathcal{M}$ is small but not exponentially small, one can just apply classical perturbation techniques to prove that it gives the first order of the splitting.

First, we give the prediction of the distance given by the Melnikov function when $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, which can be deduced from corollary 2.6. Recall that, as we have explained in section 2, in this case we study the distance between the manifolds at the section $x = 3\pi/2$. We have called this distance $\tilde{d}(\tau)$ (see (18)) and we call $\tilde{d}_0(\tau)$ the Melnikov prediction of this distance.

If $r \in (0, 2)$, calling $u^* = \ln(1 + \sqrt{2})$, the Melnikov prediction is given by

$$\tilde{d}_0(\tau) = \mu \varepsilon^{\eta-r-1} \frac{2\pi}{C} e^{-\frac{\text{Im} \rho}{\varepsilon}} (\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}(\varepsilon^{r/2}, \varepsilon^{1-r/2})), \tag{30}$$

whereas if $r = 2$, it is given by

$$\tilde{d}_0(\tau) = \mu \varepsilon^{\eta-3} \frac{2\pi e^{-\sqrt{C}}}{C^{3/2}} (\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}(\varepsilon)). \tag{31}$$

Finally, if $r > 2$, it is given by

$$\tilde{d}_0(\tau) = \mu \varepsilon^{\eta-3r/2} \frac{2\pi}{C^{3/2}} (\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}(\varepsilon^{r/2-1})). \tag{32}$$

In these cases, direct application of Melnikov theory only ensures that the distance between the manifolds is given by

$$d(\tau) = d_0(\tau) + \mathcal{O}(\mu^2 \varepsilon^{2(\eta-2r)}),$$

since the perturbation has size $\mathcal{O}(\mu\varepsilon^{\eta-2r})$. The next theorem widens the range of the validity of this prediction under certain hypotheses.

Theorem 2.10. *We consider any $\mu_0 > 0$ and we assume $\eta > \max\{r + 1, 3r/2\}$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$, $\varepsilon \in (0, \varepsilon_0)$ and $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, the invariant manifolds split and their distance on the section $x = 3\pi/2$ is given by*

- If $0 < r < 2$,

$$\tilde{d}(\tau) = \mu\varepsilon^{\eta-r-1} \frac{2\pi}{C} e^{-\frac{\text{Im } \rho_-}{\varepsilon}} (\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}(\mu\varepsilon^{\eta-r-1}, \varepsilon^{1-r/2}, \varepsilon^{r/2})).$$

- If $r = 2$,

$$\tilde{d}(\tau) = \mu\varepsilon^{\eta-3} \frac{2\pi e^{-\sqrt{C}}}{C^{3/2}} (\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}(\varepsilon, \mu\varepsilon^{\eta-3})).$$

- If $r > 2$,

$$\tilde{d}(\tau) = \mu\varepsilon^{\eta-3r/2} \frac{2\pi}{C^{3/2}} (\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}(\varepsilon^{r/2-1}, \mu\varepsilon^{\eta-3r/2})),$$

where $u^* = \ln(1 + \sqrt{2})$ and $\phi = \phi(\varepsilon, \alpha)$ is the constant given in corollary 2.6.

This theorem is proved in section 4.

3. The Melnikov function and its validity in the quasiperiodic case

Analogously to the periodic case, we work in the extended phase space that now is $(x, y, \theta_1, \theta_2) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2$. Note that we do not work in the whole symplectic space, namely we omit the actions (I_1, I_2) , since they do not have any dynamical interest. In this setting $\mathcal{T} = \{(0, 0, \theta_1, \theta_2); (\theta_1, \theta_2) \in \mathbb{T}^2\}$ is a normally hyperbolic torus and, for $\mu = 0$, the upper branches of its stable and unstable invariant manifolds coincide along the homoclinic manifold

$$\begin{aligned} \mathcal{W}^{u,+}(\mathcal{T}) = \mathcal{W}^{s,+}(\mathcal{T}) &= \{(x, y, \theta_1, \theta_2); H_0(x, y) = 0, y > 0\} \\ &= \{(x, y, \theta_1, \theta_2) = (x_0(u), y_0(u), \theta_1, \theta_2); (u, \theta_1, \theta_2) \in \mathbb{R} \times \mathbb{T}^2\}, \end{aligned}$$

where $(x_0(u), y_0(u))$ is the time-parametrization of the homoclinic orbit, which is given in (13).

Then, we look for the perturbed manifolds as

$$\begin{aligned} x &= x_0(u) \\ y &= y^{u,s}(u, \theta_1, \theta_2). \end{aligned}$$

Here $(u, \theta_1, \theta_2) \in (-\infty, U) \times \mathbb{T}^2$, for certain $U > 0$, for the unstable manifold and $(u, \theta_1, \theta_2) \in (-U, +\infty) \times \mathbb{T}^2$ for the stable one. Again, the Lagrangian character of the manifolds implies that the functions $y^{u,s}$ are given by

$$y^{u,s}(u, \theta_1, \theta_2) = \frac{1}{y_0(u)} \partial_u T^{u,s}(u, \theta_1, \theta_2).$$

Therefore, as in the periodic case, a natural way to measure the difference between the manifolds is to compute

$$D(u, \theta_1, \theta_2) = \partial_u T^s(u, \theta_1, \theta_2) - \partial_u T^u(u, \theta_1, \theta_2),$$

whose first order in μ is given by the Melnikov function

$$\mathcal{M}(u, \theta_1, \theta_2) = \int_{-\infty}^{+\infty} \{H_0, H_1\} \left(x_0(u+s), y_0(u+s), \theta_1 + \frac{s}{\varepsilon}, \theta_2 + \gamma \frac{s}{\varepsilon} \right) ds.$$

Namely, one has that

$$D(u, \theta_1, \theta_2) = \mu \varepsilon^\eta M(u, \theta_1, \theta_2) + \mathcal{O}(\mu^2 \varepsilon^{2\eta}).$$

As in the periodic case, for α small or fixed independently of ε , we measure the distance at the section $x = \pi$, which is given by

$$d(\theta_1, \theta_2) = y^s(0, \theta_1, \theta_2) - y^u(0, \theta_1, \theta_2) = \frac{1}{2} D(0, \theta_1, \theta_2), \tag{33}$$

whereas in the case $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, we measure it at the section $x = 3\pi/2$, which is given by

$$\tilde{d}(\theta_1, \theta_2) = y^s(\ln(1 + \sqrt{2}), \theta_1, \theta_2) - y^u(\ln(1 + \sqrt{2}), \theta_1, \theta_2) = 2\sqrt{2} D(\ln(1 + \sqrt{2}), \theta_1, \theta_2). \tag{34}$$

3.1. The Melnikov function

If one applies the Poincaré–Melnikov method to Hamiltonian (12), one obtains

$$\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, \alpha) = 4 \int_{-\infty}^{+\infty} \beta(u + s) F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds, \tag{35}$$

where β is the function defined in (20) and F is the quasiperiodic perturbation (9) satisfying (10) and (11).

In the quasiperiodic case, it is a well known fact that the size of the Melnikov function is not given by its first harmonic (see [Loc90, Sim94, DGJS97, DJSG99]). Instead, the leading harmonic depends on ε . For this reason, we need to compute carefully the size of all harmonics. We compute them using residuum theory and the properties of the function β given in section 2. We follow the same approach as [DGJS97]. For this reason, we first introduce the functions $c(\delta)$ defined in that paper. We consider the $2 \ln \gamma$ -periodic function

$$c(\delta) = C_0 \cosh\left(\frac{\delta - \delta_0}{2}\right) \quad \text{for } \delta \in [\delta_0 - \ln \gamma, \delta_0 + \ln \gamma], \tag{36}$$

where

$$C_0 = 2\sqrt{\frac{(\gamma r_1 + r_2)}{\gamma + \gamma^{-1}}}, \quad \delta_0 = \ln \varepsilon^*, \quad \varepsilon^* = \frac{(\gamma + \gamma^{-1})}{\gamma^2(r_1 \gamma + r_2)}$$

and it is continued by $2 \ln \gamma$ -periodicity onto the whole real axis.

One can use the function $c(\delta)$ to give the size of the Melnikov function as carried out in [DGJS97]. In the next proposition we see how this size changes depending on the relation between ε and α . In all the results in the quasiperiodic case we include the case $\alpha = 0$, since the proof we present is also valid in this case and slightly improves the results in the literature [DGJS97, Sau01].

Proposition 3.1. *We assume (10) and (11) and fix $u_0 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in [0, 1)$,*

- *If α satisfies $0 \leq \alpha \leq C\varepsilon^\nu$ where $\nu > 1$ and $C > 0$ are constants independent of ε , the Melnikov function (35) satisfies that*

$$\frac{C_1}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}} \leq \sup_{(u, \theta_1, \theta_2) \in (-u_0, u_0) \times \mathbb{T}^2} |\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, \alpha)| \leq \frac{C_2}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}} \tag{37}$$

for certain constants $u_0, C_1, C_2 > 0$.

- In the intermediate case $\alpha = \alpha_* \varepsilon + \mathcal{O}(\varepsilon^2)$, one can only give upper bounds of type

$$\sup_{(u, \theta_1, \theta_2) \in (-u_0, u_0) \times \mathbb{T}^2} |\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, \alpha)| \leq \frac{C_2}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}}$$

for certain constant $u_0, C_2 > 0$.

- If α satisfies $C\varepsilon^\nu < \alpha < 1 - C\varepsilon^r$ where $\nu \in (0, 1], r \in [0, 2)$ and $C > 0$ are constants independent of ε , the Melnikov function satisfies

$$\begin{aligned} \frac{C_1}{\sqrt{\varepsilon\alpha}(1-\alpha)^{5/4}} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}} &\leq \sup_{(u, \theta_1, \theta_2) \in (-u_0, u_0) \times \mathbb{T}^2} |\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, \alpha)| \\ &\leq \frac{C_2}{\sqrt{\varepsilon\alpha}(1-\alpha)^{5/4}} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}} \end{aligned} \tag{38}$$

for certain constants $C_1, C_2 > 0$.

Remark 3.2. Note that the definition of the function $c(\delta)$ in (36) is slightly different from the one considered in [DGJS97]. Indeed the associated constant C_0 considered in [DGJS97] includes a coefficient $\pi/2$ coming from the imaginary part of the singularity of the unperturbed separatrix. Since in the present paper the singularity changes with respect to α , we have defined a new constant which is independent of it. Then, the formulae of the splitting, see proposition 3.1, contain the dependence on the imaginary part of the singularity explicitly.

We also point out that, recalling the definition of the function $c(\delta)$ in (36), it is clear that the functions $c(\ln(2\varepsilon/\pi))$ and $c(\ln(\varepsilon/\text{Im } \rho_-))$ involved in the splitting formulae of proposition 3.1 have positive lower and upper bounds independent of ε .

Proof of proposition 3.1. As in the periodic case, to prove the first statement, instead of computing the residues of both ρ_\pm and looking for cancellations, we just expand the Melnikov integral, which is uniformly convergent as a real integral, in power series of α as

$$\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, \alpha) = \sum_{n=0}^{\infty} \alpha^n \mathcal{M}_n(u, \theta_1, \theta_2; \varepsilon), \tag{39}$$

where

$$\mathcal{M}_n(u, \theta_1, \theta_2; \varepsilon) = 4(n+1)2^n \int_{-\infty}^{+\infty} \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+3}(u+s)} F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds.$$

The function \mathcal{M}_0 was computed in [DGJS97], and in that paper it was shown that it satisfies

$$\frac{C_1}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}} \leq \sup_{(u, \theta_1, \theta_2) \in (-u_0, u_0) \times \mathbb{T}^2} |\mathcal{M}_0(u, \theta_1, \theta_2; \varepsilon, \alpha)| \leq \frac{C_2}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}}.$$

The rest of the functions \mathcal{M}_n can be bounded as follows. We Fourier-expand \mathcal{M}_n in (θ_1, θ_2) and we change the path of integration to $\text{Im } s = \pi/2 - \sqrt{\varepsilon}$ or $\text{Im } s = -(\pi/2 - \sqrt{\varepsilon})$ depending on the sign of $k \cdot \omega = k_1 + \gamma k_2$. Then, we can bound each harmonic as

$$|\mathcal{M}_n^{[k]}(u; \varepsilon)| \leq \left(\frac{K}{\varepsilon}\right)^{n+1} |F^{[k]}| e^{-\frac{|k\omega|}{\varepsilon}(\frac{\pi}{2} - \sqrt{\varepsilon})}.$$

Therefore, proceeding as in [DGJS97], one can see that $\mathcal{M}_n(u, \theta_1, \theta_2; \varepsilon)$ can be bounded as

$$|\mathcal{M}_n(u, \theta_1, \theta_2; \varepsilon)| \leq \left(\frac{K}{\varepsilon}\right)^{n+1} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}},$$

where the function c is given in (36).

Therefore, if $\alpha \ll \varepsilon$, the series (39) is convergent and the leading term is given by the first order.

For the second and third statements, we just apply the residuum theory to the Fourier harmonics of the Melnikov function (35). It can be easily seen that they have size

$$|\mathcal{M}^{[k]}(u)| = |F^{[k]}| \left| \frac{k \cdot \omega}{\varepsilon} \delta_2(\alpha) + \delta_1(\alpha) \right| e^{-\frac{|k \cdot \omega| \operatorname{Im} \rho_-}{\varepsilon}}, \tag{40}$$

where $\delta_1(\alpha)$ and $\delta_2(\alpha)$ are the functions defined in (27) and (28), respectively. For the range of α we are considering, the leading term is δ_2 , which satisfies

$$\delta_2(\alpha) \sim \frac{1}{\sqrt{\alpha}(1-\alpha)}.$$

Finally, one has to proceed as in [DGJS97]. Even if the method in that paper does not apply directly since now ρ_- depends on ε , it is enough to consider as a new parameter $q = \varepsilon/\operatorname{Im} \rho_-$, which is still small since we are assuming $C\varepsilon^\nu < \alpha < 1 - C\varepsilon^r$ where $\nu \in (0, 1]$, $r \in [0, 2)$ and $C > 0$ and therefore $\operatorname{Im} \rho_- \gg \varepsilon$. With this new parameter q , it is straightforward to obtain the size of the Melnikov function with the techniques in [DGJS97]. \square

Remark 3.3. Note that, as in the periodic case, the size of the Melnikov function depends strongly on α . When the strip of analyticity (6) is wide enough, that is $\alpha \sim \varepsilon^\nu$ with $\nu > 1$, the Melnikov function behaves as in the entire case $\alpha = 0$ [DGJS97]. In other words, the Melnikov function does not notice the finiteness of the strip of analyticity and the exponentially small coefficient is given by the imaginary part of the singularities of the separatrix. Nevertheless, note that the condition is different. In the periodic case the Melnikov function behaved as in the entire case provided $\alpha \ll \varepsilon^2$ instead of $\alpha \ll \varepsilon$.

When the strip of analyticity is independent of ε or not extremely big with respect to ε , namely taking $\alpha \sim \varepsilon^\nu$ with $\nu \in [0, 1)$, the exponentially small coefficient multiplying the periodic function $c(\delta)$ does not coincide with the imaginary part of the singularity of the unperturbed separatrix. Instead, it is given by the imaginary part of this new singularity ρ_- , which appears when one evaluates the perturbation along the unperturbed separatrix. Note that even if one takes, for instance, $\alpha = \sqrt{\varepsilon}$, which gives a strip of analyticity of order $\frac{1}{2} \ln \frac{1}{\varepsilon}$, one has that, using (21),

$$\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, \sqrt{\varepsilon}) \sim \varepsilon^{-\frac{3}{4}} e^{-c(\ln(\varepsilon/\operatorname{Im} \rho_-))\sqrt{\frac{\pi}{2\varepsilon}}\left(1 - \frac{\varepsilon^{1/4}}{\pi}\right)}.$$

That is, even for perturbations with a wide strip of analyticity with respect to ε , a correcting term appears in the exponential.

Note that the case $\alpha \sim \varepsilon$ is the boundary between these two different types of behaviour.

3.1.1. Narrow strip of analyticity: a drastic change in the size of the Melnikov function. In this section, we study how the Melnikov function increases when the analyticity strip shrinks, namely when $\alpha = 1 - C\varepsilon^r$ with $r \geq 0$ and $C > 0$. The next corollary gives upper and lower bounds for it in this case.

Corollary 3.4. *We assume (10) and (11) and fix $u_0 > 0$. Then, if one takes $\alpha = 1 - C\varepsilon^r$ with $C > 0$ and $r > 0$, the Melnikov function in (38) has the following upper and lower bounds*

- If $0 < r < 2$,

$$\begin{aligned} \frac{C_1}{\sqrt{\alpha}\varepsilon^{1/2+5r/4}} e^{-c(\ln(\varepsilon/\operatorname{Im} \rho_-))\sqrt{\frac{\operatorname{Im} \rho_-}{\varepsilon}}} &\leq \sup_{(u, \theta_1, \theta_2) \in (-u_0, u_0) \times \mathbb{T}^2} |\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, \alpha)| \\ &\leq \frac{C_2}{\sqrt{\alpha}\varepsilon^{1/2+5r/4}} e^{-c(\ln(\varepsilon/\operatorname{Im} \rho_-))\sqrt{\frac{\operatorname{Im} \rho_-}{\varepsilon}}} \end{aligned} \tag{41}$$

- If $r \geq 2$,

$$\frac{C_1}{\varepsilon^{3r/2}} \leq \sup_{(u, \theta_1, \theta_2) \in (-u_0, u_0) \times \mathbb{T}^2} |\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, 1 - C\varepsilon^r)| \leq \frac{C_2}{\varepsilon^{3r/2}}$$

for certain constants $C_1, C_2 > 0$.

The first statement of this corollary is just a rewriting of the third statement of proposition 3.1. The second one is a direct consequence of formula (40), if one takes into account the definition of δ_1 and δ_2 in (27) and (28) and the asymptotics for ρ_- in (22).

Remark 3.5. When the strip of analyticity shrinks, which occurs when α approaches 1, the size of the Melnikov function increases. Indeed, if one takes $\alpha = 1 - C\varepsilon^r$ and increasing $r \in (0, 2)$, as can be seen in (41), the Melnikov function is exponentially small but with smaller and smaller order. For instance, if one takes $\alpha = 1 - \varepsilon$, (22) shows that $\text{Im } \rho_- = \varepsilon^{1/2} + \mathcal{O}(\varepsilon)$, hence

$$\mathcal{M}(u, \theta_1, \theta_2; \varepsilon, 1 - \varepsilon) \sim \varepsilon^{-\frac{7}{4}} e^{-\frac{c(\ln(\varepsilon/\text{Im } \rho_-))}{\varepsilon^{1/4}}}.$$

The exponential order keeps decreasing until the limiting case $\alpha \sim 1 - \varepsilon^2$, then the exponential smallness breaks. In fact, when $\alpha = 1 - C\varepsilon^r$ with $r \geq 2$, the Melnikov function becomes unbounded as $\varepsilon \rightarrow 0$. As happened in the periodic case, this does not imply that the distance between the perturbed invariant manifolds blows up since, as will be proved in section 3.2.1, for $\eta > 3r/2$ this distance is given at first order by $\mu\varepsilon^\eta \mathcal{M}$ and thus is small in ε (but not exponentially small).

3.2. Validity of the Melnikov function

Once we have computed the Melnikov function in proposition 3.1, we compute the prediction it gives for the distance between the manifolds at the section $x = \pi$ for $0 < \alpha \leq \alpha_0 < 1$ for any fixed α_0

If α satisfies $0 \leq \alpha \leq C\varepsilon^\nu$ where $\nu > 1$ and $C > 0$ are constants independent of ε , we can ensure that d_0 satisfies

$$C_1 |\mu| \varepsilon^{\eta-1} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}} \leq \max_{(\theta_1, \theta_2) \in \mathbb{T}^2} |d_0(\theta_1, \theta_2)| \leq C_2 |\mu| \varepsilon^{\eta-1} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}}. \tag{42}$$

If $\alpha = \alpha_* \varepsilon + \mathcal{O}(\varepsilon^2)$ for any constant $\alpha_* > 0$, d_0 satisfies

$$\max_{(\theta_1, \theta_2) \in \mathbb{T}^2} |d_0(\theta_1, \theta_2)| \leq C_2 |\mu| \varepsilon^{\eta-1} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}}. \tag{43}$$

Finally, if $C\varepsilon^\nu < \alpha \leq \alpha_0 < 1$ where $\nu \in (0, 2)$ and $C > 0$ are constants independent of ε , d_0 satisfies

$$\begin{aligned} \frac{C_1 |\mu| \varepsilon^{\eta-\frac{1}{2}}}{\sqrt{\alpha}} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}} &\leq \max_{(\theta_1, \theta_2) \in \mathbb{T}^2} |d_0(\theta_1, \theta_2)| \\ &\leq \frac{C_2 |\mu| \varepsilon^{\eta-\frac{1}{2}}}{\sqrt{\alpha}} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}}. \end{aligned} \tag{44}$$

In all three formulae $C_1, C_2 > 0$ are constants independent of α and ε .

As we have already explained, direct application of Melnikov theory only ensures that the distance between the invariant manifolds on the section $x = \pi$ is given by

$$d(\theta_1, \theta_2) = d_0(\theta_1, \theta_2) + \mathcal{O}(\mu^2 \varepsilon^{2\eta}).$$

Therefore, the Melnikov function (35), is, in principle, first order provided μ is exponentially small with respect to ε . The next theorem shows for which range of parameters α, ε and η , the Melnikov function gives the true first order. In the quasiperiodic case, it is not known whether this range is the optimal one for which the Melnikov function predicts the splitting correctly.

Theorem 3.6. *We consider any $\mu_0 > 0$ and $\alpha_0 \in (0, 1)$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$, $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in [0, \alpha_0]$ such that $\varepsilon^{\eta-1}(\sqrt{\varepsilon} + \sqrt{\alpha})$ is small enough,*

- *If α satisfies $0 \leq \alpha \leq C\varepsilon^\nu$ where $\nu \geq 1$ and $C > 0$ are constants independent of ε , the distance between the invariant manifolds on the section $x = \pi$ is given by*

$$d(\theta_1, \theta_2) = d_0(\theta_1, \theta_2) + \mathcal{O}\left(|\mu|^2 \varepsilon^{2\eta-2} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\pi}{2\varepsilon}}}\right).$$

- *If α satisfies $C\varepsilon^\nu \leq \alpha \leq \alpha_0 < 1$ where $\nu \in [0, 1)$ and $C > 0$ are constants independent of ε , the distance between the invariant manifolds on the section $x = \pi$ is given by*

$$d(\theta_1, \theta_2) = d_0(\theta_1, \theta_2) + \mathcal{O}\left(|\mu|^2 \varepsilon^{2\eta-2} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}}\right).$$

The proof of this theorem is deferred to section 5.

Remark 3.7. Note that even if this theorem holds true provided $\varepsilon^{\eta-1}(\sqrt{\varepsilon} + \sqrt{\alpha})$ is small enough, the Melnikov prediction is true first order only if one imposes a more restrictive condition.

Comparing the size of the Melnikov prediction (42) and the size of the remainder, in the case $\alpha \ll \varepsilon$, one has to impose the condition $\eta > 1$. We observe that when $\alpha = 0$, this theorem, which gives that the Melnikov function provides the true first order of the distance if $\eta > 1$, is an improvement with respect to the sharpest previous result [DGJS97], which needed the condition $\eta > 3$ (see also [Sau01]).

In the case $\alpha \gg \varepsilon$ one has to impose $\varepsilon^{\eta-\frac{3}{2}}\sqrt{\alpha} \ll 1$.

Finally, note that the intermediate case $\alpha = \alpha_*\varepsilon + \mathcal{O}(\varepsilon^2)$ is included in the first statement of the theorem. Nevertheless, if one would check that d_0 is non-degenerate, Melnikov theory would also give the correct prediction in this case.

3.2.1. Narrow strip of analyticity: validity of the Melnikov function. In section 3.1.1 we have studied the size of the Melnikov function when $\alpha = 1 - C\varepsilon^r$ with $C, r > 0$. We devote this section to show for which range of parameters the Melnikov function gives the correct first order of the splitting.

First, we give the prediction of the distance given by the Melnikov function if $\alpha = 1 - C\varepsilon^r$ with $r > 0$ at the section $x = 3\pi/2$. If $0 < r < 2$,

$$\begin{aligned} C_1 |\mu| \varepsilon^{\eta-\frac{1}{2}-\frac{5r}{4}} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}} &\leq \max_{(\theta_1, \theta_2) \in \mathbb{T}^2} |\tilde{d}_0(\theta_1, \theta_2)| \\ &\leq C_2 |\mu| \varepsilon^{\eta-\frac{1}{2}-\frac{5r}{4}} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}} \end{aligned} \tag{45}$$

whereas if $r > 2$

$$C_1 |\mu| \varepsilon^{\eta-3r/2} \leq \max_{(\theta_1, \theta_2) \in \mathbb{T}^2} |\tilde{d}_0(\theta_1, \theta_2)| \leq C_2 |\mu| \varepsilon^{\eta-3r/2}. \tag{46}$$

The next theorem proves the validity of this prediction under certain hypotheses.

Theorem 3.8. *We consider any $\mu_0 > 0$ and $\alpha = 1 - C\varepsilon^r$ with $r > 0$, $C > 0$, and we assume $\eta > \max\{r + 1, 3r/2\}$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$ and $\varepsilon \in (0, \varepsilon_0)$, the*

invariant manifolds split and the distance between them on the section $x = 3\pi/2$ is given by

- If $0 < r < 2$,

$$\tilde{d}(\theta_1, \theta_2) = \tilde{d}_0(\theta_1, \theta_2) + \mathcal{O}\left(|\mu|^2 \varepsilon^{2\eta-2r-2} e^{-c(\ln(\varepsilon/\text{Im } \rho_-))\sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}}\right).$$

- If $r > 2$,

$$\tilde{d}(\theta_1, \theta_2) = \tilde{d}_0(\theta_1, \theta_2) + \mathcal{O}(\mu \varepsilon^{2\eta-3r}).$$

The proof of this theorem is deferred to section 5.

4. The periodic case: proof of theorem 2.8 and 2.10

In section 4.1 we prove theorem 2.8, which is when the parameter α is bounded away from 1. Nevertheless, we prove the result for a wider range of the parameter α . Therefore, our proof deals also with the first statement of theorem 2.10. For this reason, during section 4.1 we assume the condition

$$0 \leq \alpha \ll 1 - \varepsilon^2. \quad (47)$$

Note that this range of parameters corresponds to an exponentially small Melnikov function. The second statement of theorem 2.10 is proved in section 4.2

4.1. Proof of theorem 2.8

To study the splitting of separatrices we follow the approach proposed in [LMS03, Sau01, GOS10], which was inspired by Poincaré. Namely, we use the fact that the invariant manifolds are Lagrangian graphs. This allows us to look for parametrizations of the invariant manifolds as graphs of the gradient of certain generating functions.

First, we point out that we will consider τ as a complex variable to take advantage of the analyticity of the Hamiltonian with respect to this variable. To this end, we consider a fixed $\sigma > 0$ and we take

$$\tau \in \mathbb{T}_\sigma = \{\tau \in \mathbb{C}/\mathbb{Z} : |\text{Im } \tau| < \sigma\}.$$

Note that since the Hamiltonian is entire in τ , we can take any σ . From now on, all the constants appearing in this section will depend on σ . As we will see during the proof, in the periodic case, the analyticity strip with respect to τ does not play any role in the size of the splitting.

We consider the symplectic change of variables

$$\begin{aligned} x &= x_0(u) = 4 \arctan(e^u), \\ y &= \frac{w}{y_0(u)} = \frac{\cosh u}{2} w, \end{aligned} \quad (48)$$

which was introduced in [Bal06] (see also [LMS03, Sau01]), and reparametrize time as $\tau = t/\varepsilon$. With these new variables, the Hamiltonian function (5) reads

$$\varepsilon \bar{H}(u, w, \tau) = \varepsilon H\left(x_0(u), \frac{w}{y_0(u)}, \tau\right) \quad (49)$$

and the unperturbed separatrix can be parametrized as a graph as

$$w = y_0^2(u) = \frac{4}{\cosh^2 u}.$$

Then, one can look for the perturbed invariant manifolds as graphs of the gradient of generating functions $T^{u,s}(u, \tau)$. Namely, we look for functions $T^{u,s}(u, \tau)$ such that the invariant

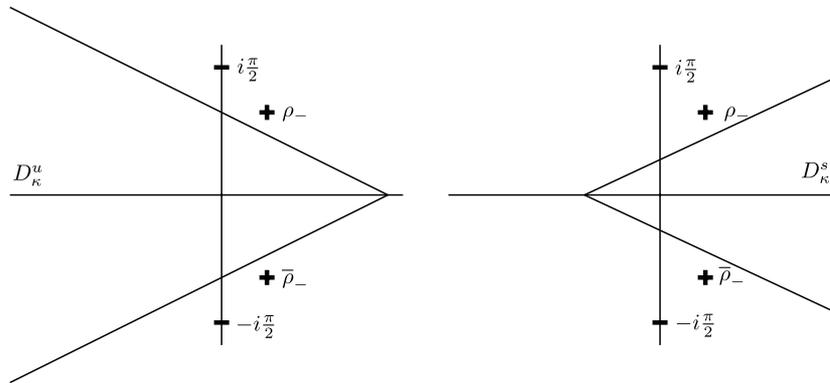


Figure 3. The domains D_κ^u and D_κ^s defined in (56).

manifolds are given by $w = \partial_u T^{u,s}(u, \tau)$. Moreover, these functions are solutions of the Hamilton–Jacobi equation

$$\varepsilon^{-1} \partial_\tau T + \overline{H}(u, \partial_u T, \tau) = 0. \tag{50}$$

This equation reads

$$\varepsilon^{-1} \partial_\tau T + \frac{\cosh^2 u}{8} (\partial_u T)^2 - \frac{4}{\cosh^2 u} + \mu \varepsilon^\eta \Psi(u) \sin \tau = 0 \tag{51}$$

with

$$\Psi(u) = \psi(x_0(u)), \tag{52}$$

where ψ is the function considered in (5) and $x_0(u)$ is the first component of the time-parametrization of the separatrix (see (48)).

Moreover, we impose the asymptotic conditions

$$\lim_{\operatorname{Re} u \rightarrow -\infty} y_0^{-1}(u) \cdot \partial_u T^u(u, \tau) = 0 \quad (\text{for the unstable manifold}), \tag{53}$$

$$\lim_{\operatorname{Re} u \rightarrow +\infty} y_0^{-1}(u) \cdot \partial_u T^s(u, \tau) = 0 \quad (\text{for the stable manifold}). \tag{54}$$

One can easily see that for $\mu = 0$, the solution of equation (51) is just

$$T_0(u) = 4 \frac{e^u}{\cosh u}, \tag{55}$$

which is the generating function of the unperturbed separatrix and has singularities at $u = \pm i\pi/2$. Nevertheless, as we have explained in section 2, the term $\Psi(u)$ has singularities at $u = \rho_\pm, \bar{\rho}_\pm$. In particular, $u = \rho_-, \bar{\rho}_-$ are closer to the real axis than $u = \pm i\pi/2, u = \rho_+$ and $u = \bar{\rho}_+$. Therefore, to study the exponentially small splitting of separatrices for this kind of system, one has to look for parametrizations of the invariant manifolds, namely solutions of equation (51), in complex domains up to a distance of order $\mathcal{O}(\varepsilon)$ of the singularities $u = \rho_-, \bar{\rho}_-$. To this end, we define the domains

$$\begin{aligned} D_\kappa^u &= \{u \in \mathbb{C}; |\operatorname{Im} u| < -\tan \beta_1 (\operatorname{Re} u - \operatorname{Re} \rho_-) + \operatorname{Im} \rho_- - \kappa \varepsilon\}, \\ D_\kappa^s &= \{u \in \mathbb{C}; |\operatorname{Im} u| < \tan \beta_1 (\operatorname{Re} u - \operatorname{Re} \rho_-) + \operatorname{Im} \rho_- - \kappa \varepsilon\}, \end{aligned} \tag{56}$$

where $\beta_1 > 0$ is an angle independent of ε and $\kappa > 0$ is such that $\operatorname{Im} \rho_- - \kappa \varepsilon > 0$ (see figure 3).

Note that this is a significant difference in comparison with the papers dealing with the exponentially small splitting of separatrices problem for algebraic or trigonometric polynomial perturbations (see [DS97, Gel97b, BF04, BFGS11]). In these cases, one needs to look for parametrizations up to a distance of order $\mathcal{O}(\varepsilon)$ of the singularities of the unperturbed separatrix. Nevertheless, in this paper, since the perturbation changes the location of the singularities of the perturbed invariant manifolds so drastically, one has to study the invariant manifolds close to these new singularities.

We recall that when α is close to 1, namely, when the analyticity strip of the Hamiltonian is very narrow, the singularity ρ_- is given by

$$\rho_- = \ln(1 + \sqrt{2}) + i(1 - \alpha)^{1/2} + \mathcal{O}(1 - \alpha).$$

Therefore, $\text{Im } \rho_- \ll \varepsilon$ if $1 - \alpha \ll \varepsilon^2$. In this case, as the Melnikov function is not exponentially small (see corollary 2.6), we can use a classical perturbative approach to prove its validity. We leave this easier yet surprising case to section 4.2. Thus, from now on, we assume condition (47) and we proceed to prove theorem 2.8. This condition ensures that the intersection $D_\kappa^u \cap D^s \kappa \cap \mathbb{R}$ contains a fundamental domain, since it is of size $\mathcal{O}(\sqrt{1 - \alpha})$. Note that if α is bounded away from 1, one can choose the angle β_1 so that the domain $D_\kappa^u \cap D^s \kappa \cap \mathbb{R}$ contains the point $u = 0$ (which corresponds to the section $x = \pi$). If $\alpha = 1 - C\varepsilon^r$ with $r \in (0, 2)$, the domain $D_\kappa^u \cap D^s \kappa \cap \mathbb{R}$ contains the point $u = \ln(1 + \sqrt{2})$ (which corresponds to the section $x = 3\pi/2$). One could change slightly the domain so that it would contain $u = 0$ also in this latter case. Nevertheless, in this case one would have to take $\beta_1 \sim \varepsilon^{r/2}$. This would lead to worse estimates which would require a stronger condition on η for the Melnikov function to be the true first order of the splitting.

The next theorem gives the existence of the invariant manifolds in the domains D_κ^* with $* = u, s$ defined in (56). We state the results for the unstable invariant manifold. The stable one has analogous properties.

Theorem 4.1. *We fix $\kappa_1 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$, $\mu \in B(\mu_0)$, if $\frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta-1}$ is small enough and (47) is satisfied, the Hamilton–Jacobi equation (50) has a unique (modulo an additive constant) real-analytic solution in $D_{\kappa_1}^u \times \mathbb{T}_\sigma$ satisfying the asymptotic condition (53).*

Moreover, there exists a real constant $b_1 > 0$ independent of ε , μ and α , such that for $(u, \tau) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma$,

$$\begin{aligned} |\partial_u T^u(u, \tau) - \partial_u T_0(u)| &\leq \frac{b_1 |\mu| \varepsilon^{\eta-1}}{(\varepsilon + \sqrt{\alpha})(1 - \alpha)}, \\ |\partial_u^2 T^u(u, \tau) - \partial_u^2 T_0(u)| &\leq \frac{b_1 |\mu| \varepsilon^{\eta-2}}{(\varepsilon + \sqrt{\alpha})(1 - \alpha)}. \end{aligned} \tag{57}$$

Furthermore, if we define the half Melnikov function

$$\mathcal{M}^u(u, \tau) = -4 \int_{-\infty}^0 \frac{\sinh(u + s) \cosh(u + s)}{(\cosh^2(u + s) - 2\alpha \sinh(u + s))^2} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds, \tag{58}$$

the generating function T^u satisfies that, for $(u, \tau) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u, \tau) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^u(u, \tau)| \leq b_1 |\mu|^2 \frac{\varepsilon^{2\eta-2}}{(1 - \alpha)^2}. \tag{59}$$

The proof of this theorem is deferred to section 4.1.1. The parametrization of the stable manifold has analogous properties. In particular, we can define

$$\mathcal{M}^s(u, \tau) = 4 \int_0^{+\infty} \frac{\sinh(u + s) \cosh(u + s)}{(\cosh^2(u + s) - 2\alpha \sinh(u + s))^2} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds, \tag{60}$$

and then, for $(u, \tau) \in D_{\kappa_1}^s \times \mathbb{T}_\sigma$,

$$|\partial_u T^s(u, \tau) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^s(u, \tau)| \leq |\mu|^2 \frac{\varepsilon^{2\eta-2}}{(1-\alpha)^2}. \tag{61}$$

Note that the Melnikov function defined in (19) is simply

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \mathcal{M}^s(u, \tau) - \mathcal{M}^u(u, \tau).$$

From now on, we omit the dependence on ε and α of the Melnikov function \mathcal{M} , which we denote by $\mathcal{M}(u, \tau)$.

Once we know the existence of parametrizations of the invariant manifolds, the next step is to study their difference. To this end, we define

$$\Delta(u, \tau) = T^s(u, \tau) - T^u(u, \tau). \tag{62}$$

This function is defined in $R_{\kappa_1} \times \mathbb{T}_\sigma$, where R_κ , for any $\kappa > 0$, is the rhomboidal domain

$$R_\kappa = D_\kappa^u \cap D_\kappa^s. \tag{63}$$

Subtracting equation (51) for both T^s and T^u , one can easily see that $\Delta \in \text{Ker} \tilde{\mathcal{L}}_\varepsilon$ for

$$\tilde{\mathcal{L}}_\varepsilon = \varepsilon^{-1} \partial_\tau + \left(\frac{\cosh^2 u}{8} (\partial_u T^s(u, \tau) + \partial_u T^u(u, \tau)) \right) \partial_u. \tag{64}$$

Since theorem 4.1 ensures that the perturbed invariant manifolds are well approximated by the unperturbed separatrix in the domains $D_{\kappa_1}^u$ and $D_{\kappa_1}^s$, we know that the operator $\tilde{\mathcal{L}}_\varepsilon$ is close to the constant coefficients operator

$$\mathcal{L}_\varepsilon = \varepsilon^{-1} \partial_\tau + \partial_u. \tag{65}$$

As is well known, any function which is defined for $(u, \tau) \in \{u \in \mathbb{C} : \text{Re } u = a, \text{Im } u \in [-r_0, r_0]\} \times \mathbb{T}_\sigma$ and belongs to the kernel of \mathcal{L}_ε , is defined in the strip $\{|\text{Im } u| < r_0\} \times \mathbb{T}_\sigma$ and has exponentially small bounds for real values of the variables. This fact is summarized in the next lemma, whose proof follows the same lines as that of the slightly different lemma 3.10 of [GOS10].

Lemma 4.2. *We consider a function $\zeta(u, \tau)$ analytic in $(u, \tau) \in \{u \in \mathbb{C} : \text{Re } u = a, \text{Im } u \in (-r_0, r_0)\} \times \mathbb{T}_\sigma$ which is a solution of $\mathcal{L}_\varepsilon \zeta = 0$. Then, ζ can be extended analytically to $\{|\text{Im } u| < r_0\} \times \mathbb{T}_\sigma$ and its mean value*

$$\langle \zeta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \zeta(u, \tau) \, d\tau$$

does not depend on u . Moreover, for $r \in (0, r_0)$ and $\sigma' \in (0, \sigma)$, we define

$$M_r = \max_{(u, \tau) \in [-ir, ir] \times \mathbb{T}_{\sigma'}} |\partial_u \zeta(u, \tau)|. \tag{66}$$

Then, provided ε is small enough, for $(u, \tau) \in \mathbb{R} \times \mathbb{T}_{\sigma'}$, the following inequality holds

$$|\partial_u \zeta(u, \tau)| \leq 4M_r e^{-\frac{r}{\varepsilon}}.$$

To apply this lemma to study the difference between the invariant manifolds, we follow [Sau01] (see also [GOS10]). Namely, we look for a change of variables which conjugates $\tilde{\mathcal{L}}_\varepsilon$ in (64) with \mathcal{L}_ε in (65).

Theorem 4.3. *We consider the constant κ_1 defined in theorem 4.1 and we fix any $\kappa_3 > \kappa_2 > \kappa_1$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying (47) and that $\varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$ is small enough, there exists a real-analytic function \mathcal{C} defined in $R_{\kappa_2} \times \mathbb{T}_\sigma$ such that the change*

$$(u, \tau) = (v + \mathcal{C}(v, \tau), \tau) \quad (67)$$

conjugates the operators $\tilde{\mathcal{L}}_\varepsilon$ and \mathcal{L}_ε defined in (64) and (65), respectively. Moreover, for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$, $v + \mathcal{C}(v, \tau) \in R_{\kappa_1}$ and there exists a constant $b_2 > 0$ such that

$$|\mathcal{C}(v, \tau)| \leq b_2 |\mu| \varepsilon^\eta \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha},$$

$$|\partial_v \mathcal{C}(v, \tau)| \leq b_2 |\mu| \varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}.$$

Furthermore, $(u, \tau) = (v + \mathcal{C}(v, \tau), \tau)$ is invertible and its inverse is of the form $(v, \tau) = (u + \mathcal{V}(u, \tau), \tau)$ where \mathcal{V} is a function defined for $(u, \tau) \in R_{\kappa_3} \times \mathbb{T}_\sigma$, which satisfies

$$|\mathcal{V}(u, \tau)| \leq b_2 |\mu| \varepsilon^\eta \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$$

and that $u + \mathcal{V}(u, \tau) \in R_{\kappa_2}$ for $(u, \tau) \in R_{\kappa_3} \times \mathbb{T}_\sigma$.

The proof of this theorem is deferred to section 4.1.2.

Once we have obtained the change of variables \mathcal{C} , we are able to prove the validity of the Melnikov prediction. We prove it by bounding $\partial_u \Delta(u, \tau) - \mathcal{M}(u, \tau)$, where Δ and \mathcal{M} are the functions defined in (62) and (19), respectively.

As a first step, we consider

$$\partial_v (\Delta(v + \mathcal{C}(v, \tau), \tau)) - \mathcal{M}(v, \tau)$$

where \mathcal{C} is the function obtained in theorem 4.3.

Theorem 4.4. *There exist $\varepsilon_0 > 0$ and $b_3 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying (47) and $\varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$ small enough, the following bound is satisfied*

$$|\partial_v (\Delta(v + \mathcal{C}(v, \tau), \tau)) - \mathcal{M}(v, \tau)| \leq b_3 |\mu|^2 \frac{\varepsilon^{2\eta-2}}{(1-\alpha)^2} e^{-\frac{\text{Im } \rho_-}{\varepsilon}}$$

for $v \in R_{\kappa_2} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$,

Proof. First, we define the Melnikov potential L (see [DG00]), namely a function such that $\partial_v L(v, \tau) = \mathcal{M}(v, \tau)$, and we study $\Delta(v + \mathcal{C}(v, \tau), \tau) - L(v, \tau)$.

As $\Delta \in \text{Ker } \tilde{\mathcal{L}}_\varepsilon$, where $\tilde{\mathcal{L}}_\varepsilon$ is the operator in (64), by theorem 4.3, the function $\Phi(v, \tau) = \Delta(v + \mathcal{C}(v, \tau), \tau) - L(v, \tau) \in \text{Ker } \mathcal{L}_\varepsilon$, where \mathcal{L}_ε is the operator defined in (65). Therefore we can apply lemma 4.2. To this end, we have to bound $\partial_v \Phi(v, \tau)$ in the domain $R_{\kappa_2} \times \mathbb{T}_\sigma$. We split Φ as $\Phi(v, \tau) = \Phi_1(v, \tau) + \Phi_2(v, \tau)$ where

$$\Phi_1(v, \tau) = \Delta(v + \mathcal{C}(v, \tau), \tau) - \Delta(v, \tau),$$

$$\Phi_2(v, \tau) = \Delta(v, \tau) - L(v, \tau).$$

where Δ is the function defined in (62) and L is the Melnikov potential.

To bound $\partial_v \Phi_1$, one has to take into account that $\Delta = (T^s - T_0) - (T^u - T_0)$ and therefore, it is enough to consider the bounds obtained in theorems 4.1 and 4.3 to obtain

$$|\partial_v \Phi_1(v, \tau)| \leq K |\mu| \varepsilon^{2\eta-2} \frac{1}{(1-\alpha)^2}$$

for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$. For the second term, it is enough to use bounds (59) and (61) to obtain

$$|\partial_v \Phi_2(v, \tau)| \leq K |\mu| \varepsilon^{2\eta-2} \frac{1}{(1-\alpha)^2}$$

for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$.

Therefore, we have that

$$|\partial_v \Phi(v, \tau)| \leq K |\mu| \varepsilon^{2\eta-2} \frac{1}{(1-\alpha)^2}$$

for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$ and then, it is enough to apply lemma 4.2 to complete the proof of theorem 4.4. \square

From this result and the exponential smallness of \mathcal{M} , given in proposition 2.1, and considering the inverse change $(v, \tau) = (u + \mathcal{V}(u, \tau), \tau)$ obtained in theorem 4.3, it is straightforward to obtain an exponentially small bound for $\partial_u \Delta(u, \tau) - \mathcal{M}(u, \tau)$. It is stated in the next corollary, whose proof is straightforward.

Corollary 4.5. *There exists $\varepsilon_0 > 0$ and $b_4 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying (47) and that $\varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$ is small enough, the following bound is satisfied:*

$$|\partial_u \Delta(u, \tau) - \mathcal{M}(u, \tau)| \leq b_4 |\mu|^2 \frac{\varepsilon^{2\eta-2}}{(1-\alpha)^2} e^{-\frac{\text{Im } \rho_-}{\varepsilon}}$$

for $u \in R_{\kappa_3} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$.

From this corollary and using that, by proposition 2.1 and corollary 2.2, the Melnikov function has non-degenerate zeros, one can see that if α is bounded away from 1 and satisfies that $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ is small enough, the manifolds intersect transversally and their distance satisfies the desired asymptotic formula. This completes the proof of theorem 2.8 (see [BFGS11]).

4.1.1. The invariant manifolds: proof of theorem 4.1. Since the proof for both invariant manifolds is analogous, we only deal with the unstable case. We look for a solution of equation (51) satisfying the asymptotic condition (53). We look for it as a perturbation of the unperturbed separatrix T_0 in (55) and therefore we work with

$$Q(u, \tau) = T(u, \tau) - T_0(u). \tag{68}$$

Replacing T in equation (51), it is straightforward to see that the equation for Q reads

$$\mathcal{L}_\varepsilon Q = \mathcal{F}(\partial_u Q, u, \tau) \tag{69}$$

where \mathcal{L}_ε is the operator defined in (65) and

$$\mathcal{F}(h, u, \tau) = -\frac{\cosh^2 u}{8} h^2 - \mu \varepsilon^\eta \Psi(u) \sin \tau. \tag{70}$$

where $\Psi(u)$ is the function defined in (52).

We devote the rest of the section to obtain a solution of equation (69) which is defined in $D_\kappa^\mu \times \mathbb{T}_\sigma$ and satisfies the asymptotic condition (53).

We start by defining a norm for functions defined in the domain D_κ^μ with $\kappa > 0$. Since we want to capture their behaviour both as $\text{Re } u \rightarrow -\infty$, namely exponential decay, and for u close to $u = ia$, we consider weighted norms with different weights. For this reason we consider $U > 0$ to divide D_κ^μ by the vertical line $\text{Re } u = -U$. Then, given $\kappa > 0$ and an analytic function $h : D_\kappa^\mu \rightarrow \mathbb{C}$, we consider

$$\begin{aligned} \|h\| = & \sup_{u \in D_\kappa^\mu \cap \{\text{Re } u < -U\}} |e^{-2u} h(u)| \\ & + \sup_{u \in D_\kappa^\mu \cap \{\text{Re } u > -U\}} |(u - \rho_-)^2 (u - \rho_+) (u - \bar{\rho}_-)^2 (u - \bar{\rho}_+) h(u)|. \end{aligned} \tag{71}$$

For analytic functions $h : D_\kappa^u \times \mathbb{T}_\sigma \rightarrow \mathbb{C}$, we consider the corresponding Fourier norm

$$\|h\|_\sigma = \sum_{k \in \mathbb{Z}} \|h^{[k]}\| e^{|k|\sigma}.$$

and the following Banach space:

$$\mathcal{E}_{\kappa,\sigma} = \{h : D_\kappa^u \times \mathbb{T}_\sigma \rightarrow \mathbb{C}; \text{ real-analytic, } \|h\|_\sigma < \infty\}. \quad (72)$$

To obtain the solutions of equation (69), we need to solve an equation of the form $\mathcal{L}_\varepsilon h = g$, where \mathcal{L}_ε is the differential operator defined in (65). Note that \mathcal{L}_ε is invertible in $\mathcal{E}_{\kappa,\sigma}$. It turns out that its inverse is \mathcal{G}_ε defined by

$$\mathcal{G}_\varepsilon(h)(u, \tau) = \int_{-\infty}^0 h(u+s, \tau + \varepsilon^{-1}s) ds. \quad (73)$$

Lemma 4.6. *The operator \mathcal{G}_ε in (73) satisfies the following properties.*

1. \mathcal{G}_ε is linear from $\mathcal{E}_{\kappa,\sigma}$ to itself, commutes with ∂_u and satisfies $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon = \text{Id}$.
2. If $h \in \mathcal{E}_{\kappa,\sigma}$, then

$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \leq K \|h\|_\sigma.$$

Furthermore, if $\langle h \rangle = 0$, then

$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \leq K\varepsilon \|h\|_\sigma.$$

3. If $h \in \mathcal{E}_{\kappa,\sigma}$, then $\partial_u \mathcal{G}_\varepsilon(h) \in \mathcal{E}_{\kappa,\sigma}$ and

$$\|\partial_u \mathcal{G}_\varepsilon(h)\|_\sigma \leq K \|h\|_\sigma.$$

Proof. It follows the same lines as the proof of lemma 5.5 in [GOS10]. \square

Once we have obtained an inverse of the operator \mathcal{L}_ε defined in (65) we can obtain solutions of equation (69) using a fixed point argument. Then, theorem 4.1 is a straightforward consequence of the following proposition.

Proposition 4.7. *We fix $\kappa_1 > 0$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying (47) and $\frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta-1}$ small enough, there exists a function Q defined in $D_{\kappa_1}^u \times \mathbb{T}_\sigma$ such that $\partial_u Q \in \mathcal{E}_{\kappa_1,\sigma}$ is a fixed point of the operator*

$$\overline{\mathcal{F}}^u(h) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(h), \quad (74)$$

where \mathcal{G}_ε and \mathcal{F} are the operators defined in (73) and (70), respectively. Furthermore, there exists a constant $b_1 > 0$ such that,

$$\begin{aligned} \|\partial_u Q\|_\sigma &\leq b_1 |\mu| \varepsilon^{\eta+1}, \\ \|\partial_u^2 Q\|_\sigma &\leq b_1 |\mu| \varepsilon^\eta. \end{aligned}$$

Moreover, if we consider the half Melnikov function defined in (58), we have that

$$\left\| \partial_u Q - \mu \varepsilon^\eta \mathcal{M}^u \right\|_\sigma \leq K |\mu|^2 \varepsilon^{2\eta} \frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha}. \quad (75)$$

Proof. We consider $\kappa_0 < \kappa_1$. It is straightforward to see that $\overline{\mathcal{F}}^u$ is well defined from $\mathcal{E}_{\kappa_0,\sigma}$ to itself. We are going to prove that there exists a constant $b_1 > 0$ such that $\overline{\mathcal{F}}^u$ sends $\overline{B}(b_1 |\mu| \varepsilon^{\eta+1}) \subset \mathcal{E}_{\kappa_0,\sigma}$ to itself and is contractive there.

We first consider $\overline{\mathcal{F}}^u(0)$. From the definition of $\overline{\mathcal{F}}^u$ in (74), the definition of \mathcal{F} in (70) and using lemma 4.6, we have that

$$\overline{\mathcal{F}}^u(0)(u, \tau) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(0)(u, \tau) = -\mu \varepsilon^\eta \mathcal{G}_\varepsilon(\Psi'(u) \sin \tau). \quad (76)$$

To bound it, first we point out that $\Psi'(u) = \beta(u)$ where $\beta(u)$ is the function defined in (20). We bound each term for $\text{Im } u \geq 0$, the other case is analogous. Using (23) and taking into account that

$$\left| \frac{u - i\pi/2}{u - \rho_+} \right| < 1, \tag{77}$$

one can easily see that $\|\Psi'(u)\| < K$. Then, taking into account that $\sin \tau$ has zero average and applying lemma 4.6, there exists a constant $b_1 > 0$ such that

$$\|\overline{\mathcal{F}}''(0)\|_\sigma \leq \frac{b_1}{2} |\mu| \varepsilon^{\eta+1}.$$

To bound the Lipschitz constant, we consider $h_1, h_2 \in \overline{B}(b_1 |\mu| \varepsilon^{\eta+1}) \in \mathcal{E}_{\kappa_0, \sigma}$. To bound $\|\overline{\mathcal{F}}''(h_2) - \overline{\mathcal{F}}''(h_1)\|_\sigma$, we need first the following bounds for $u \in D_{\kappa_0}^u$

$$\left| \frac{u - i\pi/2}{u - i\rho_-} \right| \leq 1 + \left| \frac{i\pi/2 - \rho_-}{u - i\rho_-} \right| \leq 1 + K \frac{\sqrt{\alpha}}{\varepsilon} \tag{78}$$

and

$$\left| \frac{1}{(u - i\rho_-)(u - i\overline{\rho_-})^2} \right| \leq \frac{K}{\varepsilon(1 - \alpha)}, \tag{79}$$

which are a direct consequence of (21) and (22).

Then, it is easy to see that

$$\begin{aligned} \|\overline{\mathcal{F}}''(h_2) - \overline{\mathcal{F}}''(h_1)\|_\sigma &\leq \frac{K}{\varepsilon(1 - \alpha)} \left(1 + \frac{\sqrt{\alpha}}{\varepsilon}\right) \|h_2 + h_1\|_\sigma \|h_2 - h_1\|_\sigma \\ &\leq K |\mu| \frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta-1} \|h_2 - h_1\|_\sigma. \end{aligned}$$

Then, using that by hypothesis $\frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta-1}$ is small enough,

$$\text{Lip} \overline{\mathcal{F}}'' = K |\mu| \frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta-1} < 1/2$$

and therefore $\overline{\mathcal{F}}''$ is contractive from the ball $\overline{B}(b_1 |\mu| \varepsilon^{\eta+1}) \subset \mathcal{E}_{\kappa_0, \sigma}$ into itself, and it has a unique fixed point h^* . Moreover, since it has exponential decay as $\text{Re } u \rightarrow -\infty$, we can take

$$Q(u, \tau) = \int_{-\infty}^u h^*(v, \tau) dv.$$

To obtain the bound for $\partial_u^2 Q$, it is enough to apply Cauchy estimates to the nested domains $D_{\kappa_1}^u \subset D_{\kappa_0}^u$ (see, for instance, [GOS10]), and rename b_1 if necessary.

Finally, to prove (75), it is enough to point out that $\overline{\mathcal{F}}''(0) = \mu \varepsilon^\eta \mathcal{M}''$ and consider the obtained bound for the Lipschitz constant. \square

4.1.2. Straightening the operator $\tilde{\mathcal{L}}_\varepsilon$: proof of theorem 4.3. We devote this section to prove theorem 4.3. It is a well known fact, see for instance lemma 6.3 of [GOS10], that looking for a change (67) which conjugates $\tilde{\mathcal{L}}_\varepsilon$ and \mathcal{L}_ε defined in (64) and (65) is equivalent to looking for a function \mathcal{C} solution of the equation

$$\mathcal{L}_\varepsilon \mathcal{C}(v, \tau) = \frac{\cosh^2 u}{8} \left(\partial_u T^s(u, \tau) + \partial_u T^u(u, \tau) \right) \Big|_{u=v+\mathcal{C}(v, \tau)} - 1.$$

Taking into account the definition of T_0 and Q in (55) and (68), this equation can be written as

$$\mathcal{L}_\varepsilon \mathcal{C} = \mathcal{J}(\mathcal{C}), \quad (80)$$

where

$$\mathcal{J}(h)(v, \tau) = \frac{\cosh^2 u}{8} (\partial_u \mathcal{Q}^s(u, \tau) + \partial_u \mathcal{Q}^u(u, \tau)) \Big|_{u=v+h(v, \tau)}. \quad (81)$$

To look for a solution of this equation, we start by defining some norms and Banach spaces. Given $n \in \mathbb{N}$ and a function $h : R_\kappa \rightarrow \mathbb{C}$, we define

$$\|h\|_n = \sup_{v \in R_\kappa} |(v - \rho_-)^n (v - \bar{\rho}_-)^n h(v)|. \quad (82)$$

Moreover for analytic functions $h : R_\kappa \times \mathbb{T}_\sigma \rightarrow \mathbb{C}$, we define the corresponding Fourier norm

$$\|h\|_{n, \sigma} = \sum_{k \in \mathbb{Z}} \|h^{[k]}\|_n e^{|k|\sigma}$$

and the Banach space

$$\mathcal{X}_{n, \sigma} = \{h : R_\kappa \times \mathbb{T}_\sigma \rightarrow \mathbb{C}; \text{ real-analytic, } \|h\|_{n, \sigma} < \infty\}.$$

To obtain a solution of equation (80) in the domain R_κ , we need to solve equations of the form $\mathcal{L}_\varepsilon h = g$, where \mathcal{L}_ε is the operator defined in (65). To find a right-inverse of this operator in $\mathcal{X}_{n, \sigma}$ we consider u_1 the upper vertex of R_κ and u_0 the left endpoint of R_κ . Then, we define the operator $\tilde{\mathcal{G}}_\varepsilon$ as

$$\tilde{\mathcal{G}}_\varepsilon(h)(v, \tau) = \sum_{k \in \mathbb{Z}} \tilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) e^{ik\tau}, \quad (83)$$

where its Fourier coefficients are given by

$$\begin{aligned} \tilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) &= \int_{-v_1}^v e^{i\frac{k}{\varepsilon}(w-v)} h^{[k]}(w) \, dw & \text{if } k < 0, \\ \tilde{\mathcal{G}}_\varepsilon(h)^{[0]}(v) &= \int_{v_0}^v h^{[0]}(w) \, dw \\ \tilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) &= - \int_v^{v_1} e^{i\frac{k}{\varepsilon}(w-v)} h^{[k]}(w) \, dw & \text{if } k > 0. \end{aligned}$$

The following lemma, which is proved in [GOS10] (see lemma 8.3 of this paper), gives some properties of this operator.

Lemma 4.8. *The operator $\tilde{\mathcal{G}}_\varepsilon$ in (83) satisfies the following properties.*

1. *If $h \in \mathcal{X}_{n, \sigma}$, then $\tilde{\mathcal{G}}_\varepsilon(h) \in \mathcal{X}_{n, \sigma}$ and*

$$\|\tilde{\mathcal{G}}_\varepsilon(h)\|_{n, \sigma} \leq K \|h\|_{n, \sigma}.$$

Moreover, if $\langle h \rangle = 0$,

$$\|\tilde{\mathcal{G}}_\varepsilon(h)\|_{n, \sigma} \leq K \varepsilon \|h\|_{n, \sigma}.$$

2. *If $h \in \mathcal{X}_{n, \sigma}$ with $n > 1$, then $\tilde{\mathcal{G}}_\varepsilon(h) \in \mathcal{X}_{n-1, \sigma}$ and*

$$\|\tilde{\mathcal{G}}_\varepsilon(h)\|_{n-1, \sigma} \leq \frac{K}{\sqrt{1-\alpha}} \|h\|_{n, \sigma}.$$

In the next proposition we obtain a solution of equation (80) using a fixed point argument.

Proposition 4.9. *We consider the constant $\kappa_1 > 0$ defined in theorem 4.1 and we consider any $\kappa_2 > \kappa_1$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying (47) and that $\frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta-1}$ is small enough, there exists a function $\mathcal{C} \in \mathcal{X}_{1,\sigma}$ defined in $R_{\kappa_2} \times \mathbb{T}_\sigma$ which is a fixed point of the operator*

$$\overline{\mathcal{J}}(h) = \tilde{\mathcal{G}}_\varepsilon \mathcal{J}(h), \tag{84}$$

where $\tilde{\mathcal{G}}_\varepsilon$ and \mathcal{J} are the operators defined in (83) and (81), respectively. Furthermore, $v + \mathcal{C}(v, \tau) \in R_{\kappa_1}$ for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$ and there exists a constant $b_2 > 0$ such that

$$\begin{aligned} \|\mathcal{C}\|_{1,\sigma} &\leq b_2 |\mu| \varepsilon^{\eta+1} \frac{\sqrt{\alpha} + \varepsilon}{(1 - \alpha)^{1/2}}, \\ \|\partial_v \mathcal{C}\|_{1,\sigma} &\leq b_2 |\mu| \varepsilon^\eta \frac{\sqrt{\alpha} + \varepsilon}{(1 - \alpha)^{1/2}}. \end{aligned}$$

Proof. It is straightforward to see that $\overline{\mathcal{J}}$ is well defined from $\mathcal{X}_{1,\sigma}$ to itself. We are going to prove that there exists a constant $b_2 > 0$ such that $\overline{\mathcal{J}}$ sends $\overline{B}(b_2 |\mu| \varepsilon^{\eta+1} \frac{\sqrt{\alpha} + \varepsilon}{(1 - \alpha)^{1/2}}) \subset \mathcal{X}_{1,\sigma}$ to itself and is contractive there.

We first consider $\overline{\mathcal{J}}(0)$. From the definition of $\overline{\mathcal{J}}$ in (84), the definition of \mathcal{J} in (81), we have that

$$\overline{\mathcal{J}}(0)(v, \tau) = \tilde{\mathcal{G}}_\varepsilon \mathcal{J}(0)(v, \tau) = \tilde{\mathcal{G}}_\varepsilon \left(\frac{\cosh^2 v}{8} (\partial_v Q^s(v, \tau) + \partial_v Q^u(v, \tau)) \right).$$

Using that $\partial_v Q^u$ is a fixed point of the operator $\overline{\mathcal{F}}^u$ in (74) and that $\partial_v Q^s$ is a fixed point of an analogous operator $\overline{\mathcal{F}}^s$, we can split $\overline{\mathcal{J}}(0)(v, \tau) = \mathcal{B}_1(v, \tau) + \mathcal{B}_2(v, \tau)$ with

$$\begin{aligned} \mathcal{B}_1(v, \tau) &= \tilde{\mathcal{G}}_\varepsilon \left(\frac{\cosh^2 v}{8} (\overline{\mathcal{F}}^u(0)(v, \tau) + \overline{\mathcal{F}}^s(0)(v, \tau)) \right), \\ \mathcal{B}_2(v, \tau) &= \tilde{\mathcal{G}}_\varepsilon \left(\frac{\cosh^2 v}{8} (\overline{\mathcal{F}}^u(\partial_v Q^u)(v, \tau) - \overline{\mathcal{F}}^u(0)(v, \tau) + \overline{\mathcal{F}}^s(\partial_v Q^s)(v, \tau) - \overline{\mathcal{F}}^s(0)(v, \tau)) \right). \end{aligned}$$

To bound \mathcal{B}_1 , it is enough to recall that, in the proof of proposition 4.7, we have seen that $\|\overline{\mathcal{F}}(0)\|_\sigma \leq K |\mu| \varepsilon^{\eta+1}$. Then, taking into account (77) and (78), one can see that

$$\left\| \frac{\cosh^2 v}{8} (\overline{\mathcal{F}}^u(0)(v, \tau) + \overline{\mathcal{F}}^s(0)(v, \tau)) \right\|_{1,\sigma} \leq K |\mu| \varepsilon^\eta \frac{\sqrt{\alpha} + \varepsilon}{(1 - \alpha)^{1/2}}.$$

Moreover, taking into account that $\langle \overline{\mathcal{F}}^{u,s}(0) \rangle = 0$ and applying lemma 4.8, we have that

$$\|\mathcal{B}_1\|_{1,\sigma} \leq K |\mu| \varepsilon^{\eta+1} \frac{\sqrt{\alpha} + \varepsilon}{(1 - \alpha)^{1/2}}.$$

For the second term, we first point out that

$$\overline{\mathcal{F}}^*(\partial_v Q^*)(v, \tau) - \overline{\mathcal{F}}^*(0)(v, \tau) = -\partial_v \mathcal{G}_\varepsilon \left(\frac{\cosh^2 v}{8} (\partial_v Q^*)^2 \right), \quad * = u, s.$$

Using proposition 4.7 and (77), one has that

$$\left| \frac{\cosh^2 v}{8} (\partial_v Q^*(v, \tau))^2 \right| \leq \frac{K |\mu|^2 \varepsilon^{2\eta+2}}{(v - \rho_-)^4 (v - \bar{\rho}_-)^4}.$$

Therefore, analogously to lemma 4.6, one can easily see that

$$\left| \partial_v \mathcal{G}_\varepsilon \left(\frac{\cosh^2 v}{8} (\partial_v \mathcal{Q}^*(v, \tau))^2 \right) \right| \leq \frac{K |\mu|^2 \varepsilon^{2\eta+2}}{(v - \rho_-)^4 (v - \bar{\rho}_-)^4}.$$

Using inequalities (78) and (79),

$$\left\| \frac{\cosh^2 v}{8} \left(\overline{\mathcal{F}}^*(\partial_v \mathcal{Q}^*)(v, \tau) - \overline{\mathcal{F}}^*(0)(v, \tau) \right) \right\|_{2,\sigma} \leq K |\mu|^2 \eta^{2\eta+2} \frac{(\sqrt{\alpha} + \varepsilon)^2}{1 - \alpha}.$$

Then, using lemma 4.8, one has that

$$\|\mathcal{B}_2\|_{1,\sigma} \leq K |\mu|^2 \varepsilon^{2\eta+2} \frac{(\sqrt{\alpha} + \varepsilon)^2}{(1 - \alpha)^{3/2}}.$$

Therefore, since $\varepsilon^{\eta-1} \frac{\sqrt{\alpha} + \varepsilon}{1 - \alpha} \ll 1$, there exists a constant $b_2 > 0$ such that

$$\|\overline{\mathcal{J}}(0)\|_{1,\sigma} \leq \frac{b_2}{2} |\mu| \varepsilon^{\eta+1} \frac{\sqrt{\alpha} + \varepsilon}{(1 - \alpha)^{1/2}}.$$

To bound the Lipschitz constant, it is enough to apply the mean value theorem, use the bounds of $\partial_u \mathcal{Q}^{u,s}$ and $\partial_u^2 \mathcal{Q}^{u,s}$ given in proposition 4.7 and lemma 4.8 to see that

$$\text{Lip} \leq K |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\alpha} + \varepsilon}{1 - \alpha}.$$

Then, using that $\varepsilon^{\eta-1} \frac{\sqrt{\alpha} + \varepsilon}{1 - \alpha} \ll 1$, the operator $\overline{\mathcal{J}}$ is contractive from $\overline{B}(b_2 |\mu| \varepsilon^{\eta+1} \frac{\sqrt{\alpha} + \varepsilon}{(1 - \alpha)^{1/2}}) \subset \mathcal{X}_{1,\sigma}$ to itself and it has a unique fixed point \mathcal{C} . Finally, to obtain a bound for $\partial_v \mathcal{C}$ it is enough to apply Cauchy estimates reducing slightly the domain and renaming b_2 if necessary. \square

Proof of theorem 4.3. Once we have proved proposition 4.9, it only remains to obtain the inverse change given by the function \mathcal{V} , which is straightforward using a fixed point argument. \square

4.2. Proof of theorem 2.10

The first statement of theorem 2.10 is a direct consequence of corollary 4.5 taking $\alpha = 1 - C\varepsilon^r$ with $r \in (0, 2)$. Note that the condition $\varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha}$ becomes $\eta > r + 1$. The proofs of the second and third statements, which correspond to $r \geq 2$, are considerably simpler, since we do not need to prove any exponential smallness. The first observation is that $\text{Im } \rho_- \sim \varepsilon^{r/2} \leq \varepsilon$. Even though in this case it is not necessary, we keep the analyticity properties of the parametrizations of the invariant manifolds and we work in the domains $D_\kappa^u \times \mathbb{T}_\sigma$ and $D_\kappa^s \times \mathbb{T}_\sigma$ (see (56)).

Theorem 4.10. We fix $\kappa_1 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha = 1 - C\varepsilon^r$ with $C > 0$ and $r \geq 2$, $\mu \in B(\mu_0)$, if $\eta - 3r/2 > 0$, the Hamilton–Jacobi equation (50) has a unique (modulo an additive constant) real-analytic solution in $D_{\kappa_1}^u \times \mathbb{T}_\sigma$ satisfying the asymptotic condition (53).

Moreover, there exists a real constant $b_4 > 0$ independent of ε and μ , such that for $(u, \tau) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u, \tau) - \partial_u T_0(u)| \leq b_4 |\mu| \varepsilon^{\eta-3r/2}.$$

Furthermore, for $(u, \tau) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma$, the generating function T^u satisfies that

$$|\partial_u T^u(u, \tau) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^u(u, \tau)| \leq b_4 |\mu|^2 \varepsilon^{2\eta-3r} \tag{85}$$

where \mathcal{M}^u is the function defined in (58).

Proof. The proof follows the same lines as theorem 4.1. We use the modified norm

$$\|h\| = \sup_{u \in \mathcal{D}^u \cap \{\operatorname{Re} u < -U\}} |e^{-2u}h(u)| + \sup_{u \in \mathcal{D}^u \cap \{\operatorname{Re} u > -U\}} |h(u)|. \tag{86}$$

Then, using (76), one can bound $\overline{\mathcal{F}}(0)$ as

$$\|\overline{\mathcal{F}}(0)\| = K|\mu|\varepsilon^\eta \left(K + \int_{-U}^u \frac{1}{|v - \rho_-|^2 |v - \bar{\rho}_-|^2} \right) \leq \frac{b_4}{2} |\mu|\varepsilon^{\eta-3r/2}.$$

Proceeding as before, it is straightforward to see that $\overline{\mathcal{F}}$ is contractive from the ball $\overline{B}(\mu\varepsilon^{\eta-3r/2})$ to itself with Lipschitz constant satisfying

$$\operatorname{Lip} \leq |\mu|\varepsilon^{\eta-3r/2},$$

which gives the desired result. □

The function T^s satisfies the same properties in the symmetric domain $\mathcal{D}_{\kappa_1}^s$. Note that $D_{\kappa_1}^u \cap D_{\kappa_1}^s \cap \mathbb{R}$ is an interval of size $\mathcal{O}(\varepsilon^r/2)$ centred at $u = \ln(1+\sqrt{2})$. Therefore, $D_{\kappa_1}^u \cap D_{\kappa_1}^s \cap \mathbb{R}$ does not contain a fundamental domain. Nevertheless, it suffices to deal with this domain to compute the distance between the manifolds in the section $x = 3\pi/2$.

From theorem 4.10 the formula of the distance follows. Thus, to complete the proof of theorem 2.10, it is enough to use corollary 2.6, for $r \geq 2$, to check that the Melnikov function $\mathcal{M}(\ln(1 + \sqrt{2}), \tau)$ has simple zeros so that the manifolds intersect transversally.

5. The quasiperiodic case: proof of theorems 3.6 and 3.8

As we did in the periodic case, the proof in the next section includes at the same time the results in theorem 3.6 and in the first statement of theorem 3.8. Then, section 5.2 contains the proof of the second and third statements of theorem 3.8.

5.1. Proof of theorem 3.6

We follow the same approach as in the periodic case. Therefore, we only point out the main differences with respect to it. We perform the symplectic change of variables (48) to Hamiltonian (12). In the new variables, it reads

$$\varepsilon \overline{K}(u, w, \theta_1, \theta_2, I_1, I_2) = \varepsilon K \left(x_0(u), \frac{w}{y_0(u)}, \theta_1, \theta_2, I_1, I_2 \right). \tag{87}$$

As in the periodic case, we look for the perturbed invariant manifolds as graphs of the gradient of generating functions $T^{u,s}(u, \theta_1, \theta_2)$, which are solutions of the Hamilton–Jacobi equation

$$\overline{K}(u, \partial_u T, \theta_1, \theta_2, \partial_{\theta_1} T, \partial_{\theta_2} T) = 0. \tag{88}$$

This equation reads

$$\varepsilon^{-1} \partial_{\theta_1} T + \varepsilon^{-1} \gamma \partial_{\theta_2} T + \frac{\cosh^2 u}{8} (\partial_u T)^2 - \frac{4}{\cosh^2 u} + \mu \varepsilon^\eta \Psi(u) F(\theta_1, \theta_2) = 0, \tag{89}$$

where $\Psi(u)$ is the function defined in (52).

We impose the same asymptotic conditions (53) and (54) as in the periodic case. For $\mu = 0$, the solution of equation (89) is (55). We study the existence of solutions of equation (89) in complex domains which have points close to the singularities $u = \rho_-, \bar{\rho}_-$. Nevertheless, in this case, we only need to stay at a distance of order $\mathcal{O}(\sqrt{\varepsilon})$ of the singularity instead of $\mathcal{O}(\varepsilon)$ as happened in the periodic case (see [DGJS97]). To this end, we define the modified domains

$$\begin{aligned} D_\kappa^u &= \{u \in \mathbb{C}; |\operatorname{Im} u| < -\tan \beta_1 (\operatorname{Re} u - \operatorname{Re} \rho_-) + \operatorname{Im} \rho_- - \kappa \sqrt{\varepsilon}\}, \\ D_\kappa^s &= \{u \in \mathbb{C}; |\operatorname{Im} u| < \tan \beta_1 (\operatorname{Re} u - \operatorname{Re} \rho_-) + \operatorname{Im} \rho_- - \kappa \sqrt{\varepsilon}\}. \end{aligned} \tag{90}$$

Note that the only difference with respect to the domains (56) is the change of ε by $\sqrt{\varepsilon}$.

In the quasiperiodic case, the complex domain of the angular variables plays a crucial role. As was carried out in [DGJS97, Sau01], we will prove the existence of these generating functions in very concrete domains in (θ_1, θ_2) . To this end we define the complexified torus for any $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$ with $\sigma_1, \sigma_2 > 0$,

$$\mathbb{T}_\sigma^2 = \{(\theta_1, \theta_2) \in (\mathbb{C}/\mathbb{Z})^2 : |\operatorname{Im} \theta_i| \leq \sigma_i\},$$

with

$$\sigma_i = r_i - d_i \sqrt{\varepsilon}, \tag{91}$$

where r_i are the constants defined in (10) and $d_i > 0$ are any constants independent of ε .

The next theorem gives the existence of the invariant manifolds in the domains $D_\kappa^* \times \mathbb{T}_\sigma^2$ with $*$ = u, s (see (90)). We state the results for the unstable invariant manifold. The stable one has analogous properties.

Theorem 5.1. *We fix any $\kappa_1, d_1, d_2 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$, $\mu \in B(\mu_0)$, satisfying (47) and that $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon+\sqrt{\alpha}}}{1-\alpha}$ is small enough the Hamilton–Jacobi equation (89) has a unique (modulo an additive constant) real-analytic solution in $D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$, with σ defined in (91), satisfying the asymptotic condition (53).*

Moreover, there exists a real constant $b_5 > 0$ independent of ε and μ , such that for $(u, \theta_1, \theta_2) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$,

$$\begin{aligned} |\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u)| &\leq \frac{b_5 |\mu| \varepsilon^{\eta-1}}{(\sqrt{\varepsilon} + \sqrt{\alpha})(1-\alpha)}, \\ |\partial_u^2 T^u(u, \theta_1, \theta_2) - \partial_u^2 T_0(u)| &\leq \frac{b_5 |\mu| \varepsilon^{\eta-\frac{3}{2}}}{(\sqrt{\varepsilon} + \sqrt{\alpha})(1-\alpha)}. \end{aligned}$$

Furthermore, if we define the half Melnikov function

$$\mathcal{M}^u(u, \theta_1, \theta_2) = -4 \int_{-\infty}^0 \frac{\sinh(u+s) \cosh(u+s)}{(\cosh^2(u+s) - 2\alpha \sinh(u+s))^2} F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds, \tag{92}$$

the generating function T^u satisfies that, for $(u, \theta_1, \theta_2) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$,

$$|\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^u(u, \theta_1, \theta_2)| \leq \frac{b_5 |\mu|^2 \varepsilon^{2\eta-2}}{(1-\alpha)^2}. \tag{93}$$

The proof of this theorem is deferred to section 5.1.1. The parametrization of the stable manifold has analogous properties. In particular, we can define

$$\mathcal{M}^s(u, \theta_1, \theta_2) = 4 \int_0^{+\infty} \frac{\sinh(u+s) \cosh(u+s)}{(\cosh^2(u+s) - 2\alpha \sinh(u+s))^2} F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds, \tag{94}$$

and then, for $(u, \theta_1, \theta_2) \in D_{\kappa_1}^s \times \mathbb{T}_\sigma^2$,

$$|\partial_u T^s(u, \theta_1, \theta_2) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^s(u, \theta_1, \theta_2)| \leq \frac{b_5 |\mu|^2 \varepsilon^{2\eta-2}}{(1-\alpha)^2}. \tag{95}$$

We consider the function

$$\Delta(u, \theta_1, \theta_2) = T^s(u, \theta_1, \theta_2) - T^u(u, \theta_1, \theta_2). \tag{96}$$

This function is defined in $R_\kappa \times \mathbb{T}_\sigma^2$, where R_κ is the rhomboidal domain defined by

$$R_\kappa = D_\kappa^u \cap D_\kappa^s, \tag{97}$$

where D_κ^* are the domains defined in (90).

Subtracting equation (89) for both T^s and T^u , one can easily see that $\Delta \in \text{Ker } \tilde{\mathcal{L}}_\varepsilon$ for

$$\tilde{\mathcal{L}}_\varepsilon = \varepsilon^{-1} \partial_{\theta_1} + \varepsilon^{-1} \gamma \partial_{\theta_2} + \left(\frac{\cosh^2 u}{8} (\partial_u T^s(u, \theta_1, \theta_2) + \partial_u T^u(u, \theta_1, \theta_2)) \right) \partial_u. \tag{98}$$

Since theorem 5.1 ensures that the perturbed invariant manifolds are well approximated by the unperturbed separatrix in the domains $D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$ and $D_{\kappa_1}^s \times \mathbb{T}_\sigma^2$, we know that the operator $\tilde{\mathcal{L}}_\varepsilon$ is close to the constant coefficients operator

$$\mathcal{L}_\varepsilon = \varepsilon^{-1} \partial_{\theta_1} + \varepsilon^{-1} \gamma \partial_{\theta_2} + \partial_u \tag{99}$$

in the domain $R_{\kappa_1} \times \mathbb{T}_\sigma^2$.

Any function which is defined in $\{u \in \mathbb{C}; \text{Re } u = a, \text{Im } u \in [-r_0, r_0]\} \times \mathbb{T}_\sigma^2$, for any $a \in \mathbb{R}$, and belongs to the kernel of \mathcal{L}_ε is defined in all the strip $\{|\text{Im } u| < r_0\} \times \mathbb{T}_\sigma^2$ and has exponentially small bounds for real values of the variables. This fact is summarized in the next lemma, whose proof follows the same lines as that of lemma 4.1 of [Sau01].

Lemma 5.2. *We consider a function $\zeta(u, \theta_1, \theta_2)$ analytic in $(u, \theta_1, \theta_2) \in \{u \in \mathbb{C} : \text{Re } u = \text{Re } \rho_-, |\text{Im } u| < \text{Im } \rho_- - \kappa \sqrt{\varepsilon}\} \times \mathbb{T}_\sigma^2$, where $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_i = r_i - d_i \sqrt{\varepsilon}$, which is the solution of $\mathcal{L}_\varepsilon \zeta = 0$. Then, ζ can be extended analytically to $\{|\text{Im } u| < \text{Im } \rho_- - \kappa \sqrt{\varepsilon}\} \times \mathbb{T}_\sigma^2$ and its mean value*

$$\langle \zeta \rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \zeta(u, \theta_1, \theta_2) d\theta_1 d\theta_2$$

does not depend on u . Moreover, for $\kappa' > \kappa$ and $d'_i > d_i$, we define

$$M = \max_{(u, \theta_1, \theta_2) \in [-\text{Im } \rho_- + \kappa' \sqrt{\varepsilon}, \text{Im } \rho_- - \kappa' \sqrt{\varepsilon}] \times \mathbb{T}_{\sigma'}^2} |\partial_u \zeta(u, \theta_1, \theta_2)| \tag{100}$$

where $\sigma' = (\sigma'_1, \sigma'_2)$ with $\sigma'_i = r_i - d'_i \sqrt{\varepsilon}$. Then, provided ε is small enough, for $(u, \theta_1, \theta_2) \in \mathbb{R} \times \mathbb{T}_\sigma^2$,

$$|\partial_u \zeta(u, \theta_1, \theta_2)| \leq 4M e^{-c(\ln(\varepsilon/\text{Im } \rho_-)) \sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}}.$$

where c is the periodic function defined in (36).

To apply this lemma to study the difference between the invariant manifolds, following [Sau01] (see also [GOS10]) we look for a change of variables which conjugates $\tilde{\mathcal{L}}_\varepsilon$ in (98) with \mathcal{L}_ε in (99).

Theorem 5.3. *We consider the constant $\kappa_1 > 0$ defined in theorem 4.1 and we fix any $\kappa_3 > \kappa_2 > \kappa_1$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$ and $\mu \in B(\mu_0)$ satisfying (47) and $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}$ small enough, there exists a real-analytic function \mathcal{C} defined in $R_{\kappa_2} \times \mathbb{T}_\sigma^2$ such that the change*

$$(u, \theta_1, \theta_2) = (v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2) \tag{101}$$

conjugates the operators $\tilde{\mathcal{L}}_\varepsilon$ and \mathcal{L}_ε defined in (98) and (99). Moreover, for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}_\sigma$, $v + \mathcal{C}(v, \theta_1, \theta_2) \in R_{\kappa_1}$ and there exists a constant $b_6 > 0$ such that

$$|\mathcal{C}(v, \theta_1, \theta_2)| \leq b_6 |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\alpha} + \sqrt{\varepsilon}}{1-\alpha} |\ln \varepsilon|,$$

$$|\partial_v \mathcal{C}(v, \theta_1, \theta_2)| \leq b_6 |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\alpha} + \sqrt{\varepsilon}}{1-\alpha} |\ln \varepsilon|.$$

Furthermore, $(u, \theta_1, \theta_2) = (v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2)$ is invertible and its inverse is of the form $(v, \theta_1, \theta_2) = (u + \mathcal{V}(u, \theta_1, \theta_2), \theta_1, \theta_2)$ where \mathcal{V} is a function defined for $(u, \theta_1, \theta_2) \in R_{\kappa_3} \times \mathbb{T}_\sigma^2$ and satisfies

$$|\mathcal{V}(u, \theta_1, \theta_2)| \leq b_6 |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\alpha} + \sqrt{\varepsilon}}{1-\alpha} |\ln \varepsilon|$$

and that $u + \mathcal{V}(u, \theta_1, \theta_2) \in R_{\kappa_2}$ for $(u, \theta_1, \theta_2) \in R_{\kappa_3} \times \mathbb{T}_\sigma^2$.

The proof of this theorem is deferred to section 5.1.2.

The next step is to prove the validity of the Melnikov function. To this end, we bound $\partial_u \Delta(u, \theta_1, \theta_2) - \mathcal{M}(u, \theta_1, \theta_2)$, where Δ and \mathcal{M} are the functions defined in (96) and (35), respectively. As in the periodic case, as a first step, we bound $\partial_v (\Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2)) - \mathcal{M}(v, \theta_1, \theta_2)$ where \mathcal{C} is the function obtained in theorem 5.3.

Theorem 5.4. *There exist ε_0 and $b_7 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$ and $\mu \in B(\mu_0)$, satisfying (47) and $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}$ small enough, the following bound is satisfied:*

$$|\partial_v (\Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2)) - \mathcal{M}(v, \theta_1, \theta_2)| \leq \frac{b_7 |\mu|^2 \varepsilon^{2\eta-2} |\ln \varepsilon|}{(1-\alpha)^2} e^{-c(\ln(\varepsilon/\text{Im } \rho_-)) \sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}}$$

for $v \in R_{\kappa_3} \cap \mathbb{R}$ and $(\theta_1, \theta_2) \in \mathbb{T}^2$,

Proof. First we define the Melnikov potential L , namely a function such that $\partial_u L = \mathcal{M}$ (see [DG00]). As $\Delta \in \text{Ker } \tilde{\mathcal{L}}_\varepsilon$, where $\tilde{\mathcal{L}}_\varepsilon$ is the operator in (98), by theorem 5.3, the function $\Phi(v, \theta_1, \theta_2) = \Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2) - L(v, \tau) \in \text{Ker } \mathcal{L}_\varepsilon$, where \mathcal{L}_ε is the operator defined in (99). Therefore we can apply lemma 5.2.

To this end, we have to bound $\partial_v \Phi(v, \theta_1, \theta_2)$ in the domain $R_{\kappa_2} \times \mathbb{T}_\sigma^2$. We split Φ as $\Phi(v, \theta_1, \theta_2) = \Phi_1(v, \theta_1, \theta_2) + \Phi_2(v, \theta_1, \theta_2)$ where

$$\begin{aligned} \Phi_1(v, \theta_1, \theta_2) &= \Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2) - \Delta(v, \theta_1, \theta_2), \\ \Phi_2(v, \theta_1, \theta_2) &= \Delta(v, \theta_1, \theta_2) - L(v, \theta_1, \theta_2), \end{aligned}$$

where Δ is the function defined in (96) and L is the Melnikov potential.

To bound $\partial_v \Phi_1$, one has to take into account that $\Delta = (T^s - T_0) - (T^u - T_0)$ and therefore, it is enough to consider the bounds obtained in theorems 5.1 and 5.3 to obtain

$$|\partial_v \Phi_1(v, \theta_1, \theta_2)| \leq \frac{K |\mu|^2 \varepsilon^{2\eta-2}}{(1-\alpha)^2} |\ln \varepsilon|$$

for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}_\sigma^2$. For the second term, it is enough to use bounds (95) and (93) to obtain

$$|\partial_v \Phi_2(v, \theta_1, \theta_2)| \leq \frac{K |\mu|^2 \varepsilon^{2\eta-2}}{(1-\alpha)^2} |\ln \varepsilon|$$

for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}_\sigma^2$.

Therefore, we have that

$$|\partial_v \Phi(v, \theta_1, \theta_2)| \leq \frac{K |\mu|^2 \varepsilon^{2\eta-2}}{(1-\alpha)^2} |\ln \varepsilon|$$

for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}_\sigma^2$ and then, it is enough to apply lemma 5.2 to complete the proof of theorem 5.4. \square

From this result and considering the inverse change $(v, \theta_1, \theta_2) = (u + \mathcal{V}(u, \theta_1, \theta_2), \theta_1, \theta_2)$ obtained in theorem 5.3, it is straightforward to obtain exponentially small bounds for $\partial_u \Delta(u, \theta_1, \theta_2) - \mathcal{M}(u, \theta_1, \theta_2)$ and its derivative. They are stated in the next corollary, whose proof is straightforward.

Corollary 5.5. *We consider any $\kappa_4 > \kappa_3$. Then, there exists ε_0 and $b_7 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$, satisfying (47) and $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon+\sqrt{\alpha}}}{1-\alpha}$ small enough, the following bound is satisfied:*

$$|\partial_u \Delta(u, \theta_1, \theta_2) - \mathcal{M}(u, \theta_1, \theta_2)| \leq \frac{b_7 |\mu|^2 \varepsilon^{2\eta-2} |\ln \varepsilon|}{(1-\alpha)^2} e^{-c(\ln(\varepsilon/\text{Im } \rho_-)) \sqrt{\frac{\text{Im } \rho_-}{\varepsilon}}}$$

for $u \in R_{\kappa_4} \cap \mathbb{R}$ and $(\theta_1, \theta_2) \in \mathbb{T}^2$.

This corollary completes the proof of theorem 3.6. Note that when α is bounded away from 1, we just need the simpler condition $\varepsilon^{\eta-1}(\sqrt{\varepsilon} + \sqrt{\alpha})$ small enough.

5.1.1. *The invariant manifolds: proof of theorem 5.1.* We look for a solution of equation (89) satisfying the asymptotic condition (53) as a perturbation of T_0 in (55). As in the periodic case, we define

$$Q(u, \theta_1, \theta_2) = T(u, \theta_1, \theta_2) - T_0(u), \tag{102}$$

which is a solution of

$$\mathcal{L}_\varepsilon Q = \mathcal{F}(\partial_u Q, u, \theta_1, \theta_2), \tag{103}$$

where \mathcal{L}_ε is the operator defined in (99) and

$$\mathcal{F}(h, u, \theta_1, \theta_2) = -\frac{\cosh^2 u}{8} h^2 - \mu \varepsilon^\eta \Psi(u) F(\theta_1, \theta_2), \tag{104}$$

where $\Psi(u)$ is the function defined in (52).

We devote the rest of the section to obtain a solution of equation (103), which is defined in $D_\kappa^u \times \mathbb{T}_\sigma^2$ and satisfies the asymptotic condition (53). Recall that D_κ^u has been defined in (90) and σ in (91).

We use analogous norms to the ones in the periodic case. For analytic functions $h : D_\kappa^u \times \mathbb{T}_\sigma^2 \rightarrow \mathbb{C}$, we define the Fourier norm

$$\|h\|_\sigma = \sum_{k \in \mathbb{Z}^2} \|h^{[k]}\| e^{|k_1|\sigma_1 + |k_2|\sigma_2},$$

where $\|\cdot\|$ is the norm defined in (71). We consider the Banach space

$$\mathcal{E}_{\kappa, \sigma} = \{h : D_\kappa^u \times \mathbb{T}_\sigma^2 \rightarrow \mathbb{C}; \text{real-analytic, } \|h\|_\sigma < \infty\}. \tag{105}$$

First we solve the equation $\mathcal{L}_\varepsilon h = g$, where \mathcal{L}_ε is the differential operator defined in (99). This operator is invertible in $\mathcal{E}_{\kappa, \sigma}$ and its inverse can be defined as

$$\mathcal{G}_\varepsilon(h)(u, \theta) = \int_{-\infty}^0 h(u+s, \theta_1 + \varepsilon^{-1}s, \theta_2 + \gamma \varepsilon^{-1}s) dt. \tag{106}$$

Lemma 5.6. *The operator \mathcal{G}_ε in (106) satisfies the following properties.*

1. \mathcal{G}_ε is linear from $\mathcal{E}_{\kappa, \sigma}$ to itself, commutes with ∂_u and satisfies $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon = \text{Id}$.
2. If $h \in \mathcal{E}_{\kappa, \sigma}$, then

$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \leq K \|h\|_\sigma.$$

Furthermore, one can bound each Fourier coefficient $\mathcal{G}_\varepsilon^{[k]}(h)$ with $k \neq 0$ as

$$\|\mathcal{G}_\varepsilon^{[k]}(h)\| \leq \frac{K \varepsilon}{|k \cdot \omega|} \|h^{[k]}\|.$$

3. If $h \in \mathcal{E}_{\kappa, \sigma}$, then $\partial_u \mathcal{G}_\varepsilon(h) \in \mathcal{E}_{\kappa, \sigma}$ and

$$\|\partial_u \mathcal{G}_\varepsilon(h)\|_\sigma \leq K \|h\|_\sigma.$$

Proof. The proof is analogous to the proof of lemma 5.5 of [GOS10]. □

We can obtain solutions of equation (103) using a fixed point argument. Theorem 5.1 is a straightforward consequence of the following proposition.

Proposition 5.7. *We fix $\kappa_1 > 0$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying condition (47) and $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}$ small enough, there exists a function Q defined in $D_{\kappa_1}^\mu \times \mathbb{T}_\sigma^2$ such that $\partial_u Q \in \mathcal{E}_{\kappa_1, \sigma}$ is a fixed point of the operator*

$$\overline{\mathcal{F}}^\mu(h) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(h), \tag{107}$$

where \mathcal{G}_ε and \mathcal{F} are the operators defined in (106) and (104), respectively. Furthermore, there exists a constant $b_5 > 0$ such that,

$$\begin{aligned} \|\partial_u Q\|_\sigma &\leq b_5 |\mu| \varepsilon^\eta, \\ \|\partial_u^2 Q\|_\sigma &\leq b_5 |\mu| \varepsilon^{\eta-\frac{1}{2}}. \end{aligned}$$

Moreover, if we consider the half Melnikov function defined in (92),

$$\|\partial_u Q - \mu \varepsilon^\eta \mathcal{M}^\mu\|_\sigma \leq b_5 |\mu|^2 \varepsilon^{2\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}. \tag{108}$$

Proof. We consider $\kappa_0 < \kappa_1$. It is straightforward to see that $\overline{\mathcal{F}}^\mu$ is well defined from $\mathcal{E}_{\kappa_0, \sigma}$ to itself. We are going to prove that there exists a constant $b_5 > 0$ such that $\overline{\mathcal{F}}^\mu$ sends $\overline{B}(b_5 |\mu| \varepsilon^\eta) \subset \mathcal{E}_{\kappa_0, \sigma}$ to itself and is contractive there.

We first consider $\overline{\mathcal{F}}^\mu(0)$. From the definition of $\overline{\mathcal{F}}^\mu$ in (107), the definition of \mathcal{F} in (104) and using lemma 5.6, we have that

$$\overline{\mathcal{F}}^\mu(0)(u, \theta_1, \theta_2) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(0)(u, \theta_1, \theta_2) = -\mu \varepsilon^\eta \mathcal{G}_\varepsilon(\Psi'(u)F(\theta_1, \theta_2)).$$

To bound it, we bound first each Fourier coefficient and we take advantage of the fact that $\langle F \rangle = 0$. By lemma 5.6, formula (10) and recalling that $\Psi'(u) = \beta(u)$, where β is the function defined in (20), satisfies $\|\Psi'\| \leq K$, we have that

$$\|\overline{\mathcal{F}}^\mu(0)^{[k]}\| \leq \frac{K |\mu| \varepsilon^{\eta+1}}{|k \cdot \omega|} e^{-r_1 |k_1| - r_2 |k_2|}$$

and therefore we have that

$$\|\overline{\mathcal{F}}^\mu(0)\|_\sigma \leq K |\mu| \varepsilon^{\eta+1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k \cdot \omega|} e^{-d_1 |k_1| \sqrt{\varepsilon} - d_2 |k_2| \sqrt{\varepsilon}}.$$

To bound this sum, we split it into two depending on $|k \cdot \omega| > 1/2$ or $|k \cdot \omega| < 1/2$. For the first one,

$$\sum_{\substack{k \in \mathbb{Z}^2 \setminus \{0\} \\ |k \cdot \omega| > 1/2}} \frac{1}{|k \cdot \omega|} e^{-d_1 |k_1| \sqrt{\varepsilon} - d_2 |k_2| \sqrt{\varepsilon}} \leq 2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} e^{-d_1 |k_1| \sqrt{\varepsilon} - d_2 |k_2| \sqrt{\varepsilon}} \leq \frac{K}{\varepsilon}.$$

For the second one, we take into account that, for a fixed k_2 , there exists only one k_1 satisfying $|k \cdot \omega| < 1/2$. Moreover, it satisfies

$$-|k_1| \leq -\gamma |k_2| + |k \cdot \omega| \leq -\gamma |k_2| + \frac{1}{2}.$$

Therefore, with this inequality and taking into account that for any $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$|k \cdot \omega| > \frac{C}{|k|}$$

for certain $C > 0$, we can see that

$$\sum_{\substack{k \in \mathbb{Z}^2 \setminus \{0\} \\ |k \cdot \omega| < 1/2}} \frac{1}{|k \cdot \omega|} e^{-d_1|k_1|\sqrt{\varepsilon} - d_2|k_2|\sqrt{\varepsilon}} \leq K \sum_{k_2 \in \mathbb{Z} \setminus \{0\}} |k_2| e^{-(d_1\gamma + d_2)|k_2|\sqrt{\varepsilon} + \frac{d_1}{2}\sqrt{\varepsilon}} \leq \frac{K}{\varepsilon}.$$

Then, one can easily see that there exists a constant $b_5 > 0$ such that

$$\|\overline{\mathcal{F}}^u(0)\|_\sigma \leq \frac{b_5}{2} |\mu| \varepsilon^\eta.$$

To bound the Lipschitz constant, we consider $h_1, h_2 \in \overline{B}(b_5|\mu|\varepsilon^\eta) \in \mathcal{E}_{\kappa_0, \sigma}$. Then, one can proceed as in the periodic case recalling that now

$$\left| \frac{u - i\pi/2}{u - \rho_-} \right| \leq 1 + \left| \frac{i\pi/2 - \rho_-}{u - \rho_-} \right| \leq 1 + K \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}$$

and

$$\left| \frac{1}{u - \bar{\rho}_-} \right| \leq \frac{K}{\sqrt{1 - \alpha}}.$$

Therefore,

$$\begin{aligned} \|\overline{\mathcal{F}}^u(h_2) - \overline{\mathcal{F}}^u(h_1)\|_\sigma &\leq \frac{K}{\sqrt{\varepsilon}(1 - \alpha)} \left(1 + \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}\right) \|h_2 + h_1\|_\sigma \|h_2 - h_1\|_\sigma \\ &\leq K |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} \|h_2 - h_1\|_\sigma. \end{aligned}$$

Then, using that by hypothesis $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha}$ is small enough,

$$\text{Lip} \overline{\mathcal{F}}^u = K |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} < 1/2$$

and therefore $\overline{\mathcal{F}}^u$ is contractive from the ball $\overline{B}(b_5|\mu|\varepsilon^\eta) \subset \mathcal{E}_{\kappa_0, \sigma}$ into itself, and it has a unique fixed point h^* . Moreover, since it has exponential decay as $\text{Re } u \rightarrow -\infty$, we can take

$$Q(u, \tau) = \int_{-\infty}^u h^*(v, \tau) dv.$$

To obtain the bound for $\partial_u^2 Q$, it is enough to apply Cauchy estimates to the nested domains $D_{\kappa_1}^u \subset D_{\kappa_0}^u$ (see, for instance, [GOS10]), and rename b_5 if necessary.

Finally, to prove (75), it is enough to point out that $\overline{\mathcal{F}}^u(0) = \mu \varepsilon^\eta \mathcal{M}^u$ and consider the obtained bound for the Lipschitz constant. \square

5.1.2. *Straightening the operator $\tilde{\mathcal{L}}_\varepsilon$: proof of theorem 5.3.* As we have explained in the periodic case, we just need to look for a function \mathcal{C} solution of the equation

$$\mathcal{L}_\varepsilon \mathcal{C}(v, \theta_1, \theta_2) = \frac{\cosh^2 u}{8} \left(\partial_u T^s(u, \theta_1, \theta_2) + \partial_u T^u(u, \theta_1, \theta_2) \right) \Big|_{u=v+\mathcal{C}(v, \theta_1, \theta_2)} - 1.$$

Taking into account the definition of T_0 and Q in (55) and (102), this equation can be written as

$$\mathcal{L}_\varepsilon \mathcal{C} = \mathcal{J}(\mathcal{C}) \tag{109}$$

where

$$\mathcal{J}(h)(v, \theta_1, \theta_2) = \frac{\cosh^2 u}{8} \left(\partial_u Q^s(u, \theta_1, \theta_2) + \partial_u Q^u(u, \theta_1, \theta_2) \right) \Big|_{u=v+h(v, \theta_1, \theta_2)}. \tag{110}$$

To look for a solution of this equation, we start by defining some norms and Banach spaces. Given $n \in \mathbb{N}$ and an analytic function $h : R_\kappa \times \mathbb{T}_\sigma^2 \rightarrow \mathbb{C}$, we define the Fourier norm

$$\|h\|_{n,\sigma} = \sum_{k \in \mathbb{Z}^2} \|h^{[k]}\|_n e^{|k_1|\sigma_1 + |k_2|\sigma_2},$$

where $\|\cdot\|_n$ is the norm defined in (82). Thus, we introduce the following Banach spaces:

$$\mathcal{X}_{n,\sigma} = \{h : R_\kappa \times \mathbb{T}_\sigma^2 \rightarrow \mathbb{C}; \text{ real-analytic, } \|h\|_{n,\sigma} < \infty\}.$$

To find a right-inverse of the operator \mathcal{L}_ε in (99) in $R_\kappa \times \mathbb{T}_\sigma^2$ we consider u_1 the upper vertex of R_κ and u_0 the left endpoint of R_κ . Then, we define the operator $\tilde{\mathcal{G}}_\varepsilon$ as

$$\tilde{\mathcal{G}}_\varepsilon(h)(v, \theta) = \sum_{k \in \mathbb{Z}^2} \tilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) e^{ik \cdot \theta}, \tag{111}$$

where its Fourier coefficients are given by

$$\begin{aligned} \tilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) &= \int_{-v_1}^v e^{i \frac{k \cdot \omega}{\varepsilon} (s-v)} h^{[k]}(s) ds && \text{if } k \cdot \omega < 0, \\ \tilde{\mathcal{G}}_\varepsilon(h)^{[0]}(v) &= \int_{v_0}^v h^{[0]}(s) ds, \\ \tilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) &= - \int_v^{v_1} e^{i \frac{k \cdot \omega}{\varepsilon} \omega (s-v)} h^{[k]}(s) ds && \text{if } k \cdot \omega > 0. \end{aligned}$$

The following lemma, which can be proved analogously to lemma 8.3 of [GOS10], gives some properties of this operator.

Lemma 5.8. *The operator $\tilde{\mathcal{G}}_\varepsilon$ in (111) satisfies if $h \in \mathcal{X}_{1,\sigma}$, then $\tilde{\mathcal{G}}_\varepsilon(h) \in \mathcal{X}_{0,\sigma}$ and*

$$\|\tilde{\mathcal{G}}_\varepsilon(h)\|_{0,\sigma} \leq K \frac{|\ln \varepsilon|}{\sqrt{1-\alpha}} \|h\|_{1,\sigma}.$$

Next, theorem 5.3 is a straightforward consequence of the following proposition.

Proposition 5.9. *We consider the constant $\kappa_1 > 0$ defined in theorem 5.1 and we consider any $\kappa_2 > \kappa_1$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying condition (47) and $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}$ small enough, there exists a function $\mathcal{C} \in \mathcal{X}_{1,\sigma}$ defined in $R_{\kappa_2} \times \mathbb{T}_\sigma^2$ which is a fixed point of the operator*

$$\bar{\mathcal{J}}(h) = \tilde{\mathcal{G}}_\varepsilon \mathcal{J}(h), \tag{112}$$

where $\tilde{\mathcal{G}}_\varepsilon$ and \mathcal{J} are the operators defined in (111) and (110), respectively. Furthermore, $v + \mathcal{C}(v, \theta_1, \theta_2) \in R_{\kappa_1}$ for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}_\sigma^2$ and there exists a constant $b_6 > 0$ such that

$$\begin{aligned} \|\mathcal{C}\|_{0,\sigma} &\leq b_6 |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|, \\ \|\partial_v \mathcal{C}\|_{0,\sigma} &\leq b_6 |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|. \end{aligned}$$

Proof. It is straightforward to see that $\bar{\mathcal{J}}$ is well defined from $\mathcal{X}_{1,\sigma}$ to itself. We are going to prove that there exists a constant $b_6 > 0$ such that $\bar{\mathcal{J}}$ sends $\bar{B}(b_6 |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|) \subset \mathcal{X}_{1,\sigma}$ to itself and is contractive there.

We first consider $\bar{\mathcal{J}}(0)$. From the definition of $\bar{\mathcal{J}}$ in (112), the definition of \mathcal{J} in (110), we have that

$$\bar{\mathcal{J}}(0)(v, \theta_1, \theta_2) = \tilde{\mathcal{G}}_\varepsilon \mathcal{J}(0)(v, \theta_1, \theta_2) = \tilde{\mathcal{G}}_\varepsilon \left(\frac{\cosh^2 v}{8} (\partial_v Q^s(v, \theta_1, \theta_2) + \partial_v Q^u(v, \theta_1, \theta_2)) \right).$$

Then, it is enough to apply lemma 5.8 and proposition 5.7 to see that there exists a constant $b_6 > 0$ such that

$$\|\overline{\mathcal{J}}(0)\|_{0,\sigma} \leq \frac{b_6}{2} |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|.$$

To bound the Lipschitz constant, it is enough to apply the mean value theorem, use the bound of $\partial_u^2 Q$ of proposition 5.7 and lemma 5.8 to see that

$$\text{Lip} \leq K |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|.$$

Then, using that $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha} |\ln \varepsilon| \ll 1$, the operator $\overline{\mathcal{J}}$ is contractive from $\overline{B}(b_6 |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|) \subset \mathcal{X}_{1,\sigma}$ to itself and it has a unique fixed point \mathcal{C} . Finally, to obtain a bound for $\partial_v \mathcal{C}$ it is enough to apply Cauchy estimates reducing slightly the domain and renaming b_6 if necessary. \square

Proof of theorem 5.3. Once we have proved proposition 5.9, it only remains to obtain the inverse change given by the function \mathcal{V} , which is straightforward using a fixed point argument. \square

5.2. Proof of theorem 3.8

The first statement of theorem 3.8 is a direct consequence of corollary 5.5 taking $\alpha = 1 - C\varepsilon^r$ with $r \in (0, 2)$. Note that the condition $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}$ becomes, as in the periodic case, $\eta > r + 1$. The proofs of the second and third statements, which correspond to $r \geq 2$, are considerably simpler, since we do not need to prove any exponential smallness. It follows the same lines as the proof of theorem 2.10.

Theorem 5.10. We fix $\kappa_1 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha = 1 - C\varepsilon^r$ with $C > 0$ and $r \geq 2$, $\mu \in B(\mu_0)$, if $\eta - 3r/2 > 0$, the Hamilton–Jacobi equation (88) has a unique (modulo an additive constant) real-analytic solution in $\mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma^2$ satisfying the asymptotic condition (53).

Moreover, there exists a real constant $b_8 > 0$ independent of ε and μ , such that for $(u, \theta_1, \theta_2) \in \mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma^2$,

$$|\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u)| \leq b_8 |\mu| \varepsilon^{\eta-3r/2}.$$

Furthermore, for $(u, \theta_1, \theta_2) \in \mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma^2$, the generating function T^u satisfies that

$$|\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^u(u, \theta_1, \theta_2)| \leq b_8 |\mu|^2 \varepsilon^{2\eta-3r}, \tag{113}$$

where \mathcal{M}^u is the function defined in (92).

Proof. It follows the same lines as the proof of theorem 4.10. We use the modified norm (86) and we bound $\overline{\mathcal{F}}(0)$ as

$$\|\overline{\mathcal{F}}(0)\| = K |\mu| \varepsilon^\eta \left(K + \int_{-U}^u \frac{1}{|v - \rho_-|^2 |v - \bar{\rho}_-|^2} \right) \leq \frac{b_4}{2} |\mu| \varepsilon^{\eta-3r/2}.$$

Finally, it is straightforward to see that $\overline{\mathcal{F}}$ is contractive from the ball $\overline{B}(\mu \varepsilon^{\eta-3r/2})$ to itself with Lipschitz constant equal to

$$\text{Lip} \leq |\mu| \varepsilon^{\eta-3r/2},$$

which gives the desired result. \square

The function T^s satisfies the same properties in the symmetric domain $\mathcal{D}_{\kappa_1}^s$. From this theorem, the formula of the distance $\tilde{d}(\theta_1, \theta_2)$ follows.

Acknowledgments

The authors want to thank the referee for his/her help to improve the final version of the manuscript. The authors have been partially supported by the Spanish MCyT/FEDER grant MTM2009-06973 and the Catalan SGR grant 2009SGR859. In addition, the research of MG has been supported by the Spanish PhD grant FPU AP2005-1314. Part of this work was carried out while MG stayed at the Pennsylvania State University and at the Fields Institute. He wants to thank these institutions for their hospitality and support.

Appendix. Some remarks on the singular periodic case

We devote this section to give some remarks and conjectures about the singular case, namely when the Melnikov function does not predict correctly the splitting of separatrices. We restrict this discussion to the case $0 < \alpha \leq \alpha_0 < 1$ for any fixed α_0 , where theorem 2.8 holds.

The main point in the proof of the exponentially small splitting of separatrices is to give a good approximation of the perturbed invariant manifolds not only in the real line but in a complex strip which reaches a neighbourhood of order $\mathcal{O}(\varepsilon)$ of the singularity $u = \rho_-$ of the function (20). In this paper, this is the result given in theorem 4.1. To prove this theorem, one has to impose the condition $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ small enough (see also remark 2.9). Looking at the first order in μ of the perturbed invariant manifolds, which is given by the half Melnikov functions \mathcal{M}^s and \mathcal{M}^u (see (58) and (60)), one can see that this condition is necessary. Indeed, when $u - \rho_- \sim \varepsilon$ the size of \mathcal{M}^s and \mathcal{M}^u is of the same order as the error term (57) in theorem 4.1, which is of size $\frac{\varepsilon^{\eta-1}}{(\varepsilon + \sqrt{\alpha})}$. Moreover, the difference between these first order approximations \mathcal{M}^s and \mathcal{M}^u and the perturbed invariant manifolds is given in (59) and (61), and it is of order $\mu^2 \varepsilon^{2\eta-2}$ for $u - \rho_- \sim \varepsilon$. Therefore, if $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ is of order 1, the half Melnikov functions \mathcal{M}^s and \mathcal{M}^u have the same size as the remainder. This implies that, when $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ is not small, at a distance $\mathcal{O}(\varepsilon)$ of the singularities of the function (20) the functions \mathcal{M}^s and \mathcal{M}^u are not a good approximation of the perturbed invariant manifolds. Then, in these cases the Melnikov integral, which is just $\mathcal{M} = \mathcal{M}^s - \mathcal{M}^u$, fails to predict correctly the splitting of separatrices. This is usually called the singular case. The correct approach to deal with it is to look for the first order of the invariant manifolds at a distance $\mathcal{O}(\varepsilon)$ of the singularities in a different way. As was first pointed out in [Laz84] in the study of the Standard Map, these new first orders are solutions of a different equation usually called the *inner equation*, which is independent of ε (see [Gel97b, Gel00, GS01, BS08, MSS10, GG11]). For classical Hamiltonian systems, the inner equation is a new Hamilton–Jacobi equation, which has been studied in several models in [OSS03, Bal06, GOS10, BFGS11]. Nevertheless, in all these works, the inner equation was considered for points at a distance $\mathcal{O}(\varepsilon)$ of the singularity of the unperturbed separatrix since it was also the singularity of the corresponding function β in (20), and this singularity gave the exponentially small coefficient in the splitting. Nevertheless, for system (4) one has to proceed more carefully since the singularities of β , and therefore also the exponentially small coefficient, depend on α (see proposition 2.1 and theorem 2.8). Consequently, we will need to look for different inner equations depending on the relation between α and ε . Namely, in some cases ($0 < \alpha \leq \varepsilon^2$, that is when the singularities of β and the singularities of the unperturbed homoclinic are closer than ε) we will study the inner equation close to $u = i\pi/2$ and in others ($\varepsilon^2 \ll \alpha \leq \alpha_0 < 1$) close to $u = \rho_-$.

We start with the case $0 < \alpha \ll \varepsilon^2$, which corresponds to a wide analyticity strip. As explained in remark 2.9, for this range of parameters the singular case is $\eta = 0$. In proposition 2.1 we have seen that the exponential coefficient in the Melnikov function is given

by the imaginary part of the singularity of the unperturbed separatrix. Namely, the analyticity strip is so wide that the Melnikov function behaves as in the entire case $\alpha = 0$. Then, to study the singular case $\eta = 0$, one has to look for good approximations of the invariant manifolds in a neighbourhood of order $\mathcal{O}(\varepsilon)$ of $u = i\pi/2$. To this end, one has to perform the singular change $u = i\pi/2 + \varepsilon z$. Moreover, since in theorem 4.1 we have seen that at a distance $\mathcal{O}(\varepsilon)$ of $u = i\pi/2$ the generating functions $T^{u,s}$ satisfy $T^{u,s} \sim 1/\varepsilon$, we define the scaled generating functions $\varphi^{u,s}(z, \tau) = \varepsilon T^{u,s}(i\pi/2 + \varepsilon z, \tau)$ (see [OSS03]). Since the generating functions $T^{u,s}$ were solutions of the Hamilton–Jacobi equation (51), one can apply the just explained rescalings to this equation to obtain the Hamilton–Jacobi equation satisfied by the functions $\varphi^{u,s}$. If we let $\varepsilon \rightarrow 0$ in this equation we obtain the inner equation

$$\partial_\tau \varphi_0 - \frac{z^2}{8} (\partial_z \varphi_0)^2 + \frac{2}{z^2} (1 - \mu \sin \tau) = 0, \tag{114}$$

which depends neither on ε nor on α .

Certain solutions $\varphi_0^{u,s}$ of this equation are the candidates to be the first order of the functions $\varphi^{u,s}$, namely the first order of the parametrizations of the invariant manifolds close to the singularity. Then, the study of their difference would give the first order of the difference between the invariant manifolds. This equation was already studied in [OSS03] using resurgence theory and in [Bal06] using classical functional analysis techniques.

The case $\alpha \sim \varepsilon^2$ is the transition case for which, as for $0 < \alpha \ll \varepsilon^2$, the singular case is $\eta = 0$ (see remark 2.9). In proposition 2.1 we have seen that the exponential coefficient is given by $\pi/2$ but the residua of both $u = \rho_\pm$ make a contribution to the Melnikov function. We assume that

$$\alpha(\varepsilon) = \alpha_* \varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

Then, proceeding as in the previous case, the natural change to inner variable is given by $u = i\pi/2 + \varepsilon z$ and the rescaling in the generating function by $\varphi^{u,s}(z, \tau) = \varepsilon T^{u,s}(i\pi/2 + \varepsilon z, \tau)$. Taking into account (21), applying the rescalings to (51) and letting $\varepsilon \rightarrow 0$, we obtain the inner equation

$$\partial_\tau \varphi_0 - \frac{z^2}{8} (\partial_z \varphi_0)^2 + \frac{2}{z^2} + \mu \frac{i}{2(z^2 + 2i\alpha_*)} \sin \tau = 0.$$

Finally we deal with the case $\alpha \gg \varepsilon^2$. We first consider the case α independent of ε , and from it we will deduce the case $\alpha \sim \varepsilon^v$ with $v \in (0, 2)$. If we take a fixed α , we have seen that one has to study the parametrizations of the invariant manifolds close to $u = \rho_-$ and that the singularity $u = \rho_+$ does not play any role in the size of the splitting. Note that, as explained in remark 2.9, now the limiting case is $\eta = 1$, that is, we deal with equation

$$\varepsilon^{-1} \partial_\tau T + \frac{\cosh^2 u}{8} (\partial_u T)^2 - \frac{4}{\cosh^2 u} + \mu \varepsilon \Psi(u) \sin \tau = 0$$

(see equation (51)). As a first step, we can expand the parametrization of the invariant manifold $T^{u,s}(u, \tau)$ as a power series of ε . It can be easily seen that

$$T^{u,s}(u, \tau) \sim \sum_{k \geq 0} \varepsilon^k T_k(u, \tau)$$

where T_0 corresponds to the separatrix (55). One can see that the terms of the series satisfy that for $k \geq 0$,

$$\partial_u T_k(u, \tau) \sim \frac{1}{(u - \rho_-)^k}$$

and therefore they all become of the same size at a distance ε of the singularity. Nevertheless, in this case one has to be more careful, since if one considers the asymptotic size of the power series terms of the generating function T instead of $\partial_u T$, we have that

$$T_0(u) \sim 1 \quad \text{and} \quad T_k(u, \tau) \sim \frac{1}{(u - \rho_-)^{k-1}} \quad \text{for } k \geq 1.$$

Namely, at a distance of order $\mathcal{O}(\varepsilon)$ of $u = \rho_-$ all the terms with $k \geq 1$ become of the same order but T_0 is still bigger. Therefore, it is more convenient to deal with the function $Q = T - T_0$. Now, one can consider the change to inner variable $u = \rho_- + \varepsilon z$. Moreover, since for u such that $u - \rho_- \sim \varepsilon$ one has that $\partial_u Q \sim 1$, one has to rescale the generating function as $\phi(z, \tau) = \varepsilon^{-1} Q(\rho_- + \varepsilon z, \tau)$. This change leads to the inner equation

$$\partial_\tau \phi + \partial_z \phi + \frac{\cosh^2 \rho_-}{8} (\partial_z \phi)^2 - \mu \frac{\delta_2(\alpha)}{z} \sin \tau = 0,$$

where $\delta_2(\alpha)$ is the function defined in (28).

Proceeding analogously, one can deduce the inner equation for the case $\alpha = \alpha_* \varepsilon^\nu$ with $\nu \in (0, 2)$. Recall that for this range of α the singular case was $\eta = 1 - \nu/2$ (see remark 2.9). Namely, we deal with the following equation in $Q(u, \tau) = T(u, \tau) - T_0(u)$:

$$\varepsilon^{-1} \partial_\tau Q + \partial_u Q + \frac{\cosh^2 u}{8} (\partial_u Q)^2 + \mu \varepsilon^{1-\frac{\nu}{2}} \Psi(u) \sin \tau = 0.$$

As in the previous case, we study the inner equation close to $u = \rho_-$ and therefore the change to inner variable is still $u = \rho_- + \varepsilon z$. Nevertheless, now the size of the generating function Q at a distance $\mathcal{O}(\varepsilon)$ of $u = \rho_-$ is $Q \sim \varepsilon^{1-\nu}$ and therefore the suitable rescaling for Q is given by $\phi(z, \tau) = \varepsilon^{-(1-\nu)} Q(\rho_- + \varepsilon z, \tau)$. Then, proceeding as before and taking into account the definition of $\delta_2(\alpha)$ in (28), one can obtain the following inner equation:

$$\partial_\tau \phi + \partial_z \phi - \frac{i\alpha_*}{4} (\partial_z \phi)^2 + \frac{\mu}{(1+i)\sqrt{\alpha_*}} \frac{1}{z} \sin \tau = 0.$$

We expect that studying all these inner equations, one could obtain the true first asymptotic order of the difference between the perturbed invariant manifolds.

References

- [Arn64] Arnold V I 1964 Instability of dynamical systems with several degrees of freedom *Sov. Math. Dokl.* **5** 581–5
- [Bal06] Baldomá I 2006 The inner equation for one and a half degrees of freedom rapidly forced Hamiltonian systems *Nonlinearity* **19** 1415–45
- [Ber08] Bernard P 2008 The dynamics of pseudographs in convex Hamiltonian systems *J. Am. Math. Soc.* **21** 615–69
- [BF04] Baldomá I and Fontich E 2004 Exponentially small splitting of invariant manifolds of parabolic points *Mem. Am. Math. Soc.* **167** x–83
- [BF05] Baldomá I and Fontich E 2005 Exponentially small splitting of separatrices in a weakly hyperbolic case *J. Diff. Eqns* **210** 106–34
- [BFGS11] Baldomá I, Fontich E, Guardia M and Seara T M 2011 Exponentially small splitting of separatrices beyond Melnikov analysis: rigorous results *Preprint arXiv:1201.5152*
- [Bou10] Bounemoura A 2010 Nekhoroshev estimates for finitely differentiable quasi-convex Hamiltonians *J. Diff. Eqns* **249** 2905–20
- [BS08] Baldomá I and Seara T M 2008 The inner equation for generic analytic unfoldings of the Hopf-zero singularity *Discrete Contin. Dyn. Syst. B* **10** 323–47
- [CG94] Chierchia L and Gallavotti G 1994 Drift and diffusion in phase space *Ann. Inst. H. Poincaré Phys. Théor.* **60** 144
- [CY04] Cheng C Q and Yan J 2004 Existence of diffusion orbits in *a priori* unstable Hamiltonian systems *J. Diff. Geom.* **67** 457–517

- [DdILS06] Delshams A, de la Llave R and Seara T M 2006 A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model *Mem. Am. Math. Soc.* **179** 141
- [DG96] Delshams A and Gutiérrez P 1996 Effective stability and KAM theory *J. Diff. Eqns* **128** 415–90
- [DG00] Delshams A and Gutiérrez P 2000 Splitting potential and the Poincaré–Melnikov method for whiskered tori in Hamiltonian systems *J. Nonlinear Sci.* **10** 433–76
- [DG03] Delshams A and Gutiérrez P 2003 Exponentially small splitting of separatrices for whiskered tori in Hamiltonian systems *Zap. Nauchn. Sem. St-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 300 (*Teor. Predst. Din. Sist. Spets. Vyp.* **8** 87–121, 287)
- [DGJS97] Delshams A, Gelfreich V, Jorba À and Seara T M 1997 Exponentially small splitting of separatrices under fast quasiperiodic forcing *Commun. Math. Phys.* **189** 35–71
- [DGS04] Delshams A, Gutiérrez P and Seara T M 2004 Exponentially small splitting for whiskered tori in Hamiltonian systems: flow-box coordinates and upper bounds *Discrete Contin. Dyn. Syst.* **11** 785–826
- [DH09] Delshams A and Huguet G 2009 Geography of resonances and Arnold diffusion in *a priori* unstable Hamiltonian systems *Nonlinearity* **22** 1997–2077
- [DJSG99] Delshams A, Jorba À, Seara T M and Gelfreich V 1999 Splitting of separatrices for (fast) quasiperiodic forcing *Hamiltonian Systems with Three or More Degrees of Freedom (S'Agaró, 1995) (NATO Advanced Science Institute Series C: Mathematical and Physical Sciences* vol 533) (Dordrecht: Kluwer) pp 367–71
- [DS92] Delshams A and Seara T M 1992 An asymptotic expression for the splitting of separatrices of the rapidly forced pendulum *Commun. Math. Phys.* **150** 433–63
- [DS97] Delshams A and Seara T M 1997 Splitting of separatrices in Hamiltonian systems with one and a half degrees of freedom *Math. Phys. Electron. J.* **3** Paper 4, 40pp (electronic)
- [FGKR11] Féjóz J, Guardia M, Kaloshin V and Roldán P 2011 Diffusion along mean motion resonance in the restricted planar three body problem *Preprint* arXiv:1109.2892
- [Fon93] Fontich E 1993 Exponentially small upper bounds for the splitting of separatrices for high frequency periodic perturbations *Nonlinear Anal.* **20** 733–44
- [Fon95] Fontich E 1995 Rapidly forced planar vector fields and splitting of separatrices *J. Diff. Eqns* **119** 310–35
- [Gel94] Gelfreich V G 1994 Separatrices splitting for the rapidly forced pendulum *Seminar on Dynamical Systems (St Petersburg, 1991) (Progress in Nonlinear Differential Equations and their Applications* vol 12) (Basel: Birkhäuser) pp 47–67
- [Gel97a] Gelfreich V G 1997 Melnikov method and exponentially small splitting of separatrices *Physica D* **101** 227–48
- [Gel97b] Gelfreich V G 1997 Reference systems for splittings of separatrices *Nonlinearity* **10** 175–93
- [Gel00] Gelfreich V G 2000 Separatrix splitting for a high-frequency perturbation of the pendulum *Russ. J. Math. Phys.* **7** 48–71
- [GG11] Gaivão J P and Gelfreich V 2011 Splitting of separatrices for the Hamiltonian–Hopf bifurcation with the Swift–Hohenberg equation as an example *Nonlinearity* **24** 677–98
- [GGM99] Gallavotti G, Gentile G and Mastropietro V 1999 Separatrix splitting for systems with three time scales *Commun. Math. Phys.* **202** 197–236
- [GH83] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Berlin: Springer)
- [GOS10] Guardia M, Olivé C and Seara T 2010 Exponentially small splitting for the pendulum: a classical problem revisited *J. Nonlinear Sci.* **20** 595–685
- [GS01] Gelfreich V and Sauzin D 2001 Borel summation and splitting of separatrices for the Hénon map *Ann. Inst. Fourier (Grenoble)* **51** 513–67
- [HMS88] Holmes P, Marsden J and Scheurle J 1988 Exponentially small splittings of separatrices with applications to KAM theory and degenerate bifurcations *Hamiltonian Dynamical Systems (Contemporary Mathematics* vol 81)
- [Laz84] Lazutkin V F 1984 Splitting of separatrices for the Chirikov standard map. VINITI 6372/82 Preprint (in Russian)
- [LMS03] Lochak P, Marco J-P and Sauzin D 2003 On the splitting of invariant manifolds in multidimensional near-integrable Hamiltonian systems *Mem. Am. Math. Soc.* **163** viii+145
- [Loc90] Lochak P 1990 Effective speed of Arnold's diffusion and small denominators *Phys. Lett. A* **143** 39–42
- [LS80] Llibre J and Simó C 1980 Oscillatory solutions in the planar restricted three-body problem *Math. Ann.* **248** 153–84

- [Mel63] Melnikov V K 1963 On the stability of the center for time periodic perturbations *Trans. Moscow Math. Soc.* **12** 1–57
- [MP94] Martínez R and Pinyol C 1994 Parabolic orbits in the elliptic restricted three body problem *J. Diff. Eqns* **111** 299–339
- [MS02] Marco J P and Sauzin D 2002 Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems *Publ. Math. Inst. Hautes Études Sci.* **96** 199–275
- [MSS10] Martín P, Sauzin D and Seara T M 2011 Resurgence of inner solutions for perturbations of the McMillan map *Discrete Contin. Dyn. Syst.* **31** 165–207
- [Nek77] Nekhoroshev N N 1977 An exponential estimate of the time of stability of nearly integrable Hamiltonian systems *Usp. Mat. Nauk* **32** 5–66, 287
- [Oli06] Olivé C 2006 Càlcul de l'escissió de separatrius usant tècniques de matching complex i ressurgència aplicades a l'equació de Hamilton–Jacobi <http://www.tdx.cat/TDX-0917107-125950>
- [OSS03] Olivé C, Sauzin D and Seara T M 2003 Resurgence in a Hamilton–Jacobi equation *Proc. Int. Conf. in Honor of Frédéric Pham (Nice, France, 2002)*, *Ann. Inst. Fourier* **53** 1185–235
- [Poi99] Poincaré H 1892–1899 *Les Méthodes Nouvelles de la Mécanique Céleste* vols 1, 2, 3 (Paris: Gauthier-Villars)
- [Pös93] Pöschel J 1993 Nekhoroshev estimates for quasi-convex Hamiltonian systems *Math. Z.* **213** 187–216
- [Sau95] Sauzin D 1995 Résurgence paramétrique et exponentielle petite de l'écart des séparatrices du pendule rapidement forcé *Ann. Inst. Fourier* **45** 453–511
- [Sau01] Sauzin D 2001 A new method for measuring the splitting of invariant manifolds *Ann. Sci. École Norm. Sup. (4)* **34** 159–221
- [Sim94] Simó C 1994 Averaging under fast quasiperiodic forcing *Hamiltonian Mechanics (Toruń, 1993)* (NATO Advanced Science Institute Series B: Physics vol 331) (New York: Plenum) pp 13–34
- [SMH91] Scheurle J, Marsden J E and Holmes P 1991 Exponentially small estimates for separatrix splittings *Asymptotics Beyond all Orders (La Jolla, CA, 1991)* (NATO Advanced Science Institute Series B: Physics vol 284) (New York: Plenum) pp 187–95
- [SV01] Simó C and Valls C 2001 A formal approximation of the splitting of separatrices in the classical Arnold's example of diffusion with two equal parameters *Nonlinearity* **14** 1707–60
- [Tre97] Treschev D 1997 Separatrix splitting for a pendulum with rapidly oscillating suspension point *Russ. J. Math. Phys.* **5** 63–98
- [Tre04] Treschev D 2004 Evolution of slow variables in *a priori* unstable Hamiltonian systems *Nonlinearity* **17** 1803–41
- [Xia92] Xia Z 1992 Melnikov method and transversal homoclinic points in the restricted three-body problem *J. Diff. Eqns* **96** 170–84