

**An instability mechanism along the mean motion  
resonances in the restricted three body problem**

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- We consider the **Restricted Planar Elliptic 3 Body Problem**.
- Namely, we study the motion  $q(t)$  of a massless body (Asteroid) under the influence of two primaries  $q_1(t)$  and  $q_2(t)$  of masses  $\mu$  and  $1 - \mu$ , which move along ellipses of eccentricity  $e_0 > 0$  around their center of mass.
- We consider
  - $\mu = 10^{-3}$  which is a realistic value for the Sun-Jupiter model.
  - $e_0 > 0$  arbitrarily small.

## The 2 Body Problem

- If we omit the influence of Jupiter ( $\mu = 0$ ), the system is reduced to two uncoupled 2 Body Problems (Sun-Jupiter and Sun-Asteroid).
- The motion of the Asteroid is given by Kepler Laws.
- First Kepler Law: Orbits of the 2BP are conic sections
- Assume that the Asteroid is moving along an ellipse.
- An ellipse can be given by its semimajor axis  $a$  and its eccentricity  $0 < e < 1$ .
- For the 2BP these parameters are constants of motion.

## The mean motion resonances

- Third Kepler Law: Period of motion of the ellipse is  $2\pi a^{3/2}$  where  $a$  is the semimajor axis of the ellipse.
- **Mean motion resonance** is resonance between the period of the Asteroid and the period of Jupiter
- If we normalize the period of Jupiter to  $2\pi$  (and its semimajor axis to 1), mean motion resonance appears when  $a^{3/2}$  is rational.

- We want to see the influence of Jupiter ( $\mu = 10^{-3}$ ) on the shape of the ellipse when the Asteroid is in mean motion resonance.
- We have focused our study in the mean motion resonance 1 : 7 (period of the Asteroid is seven times the period of Jupiter).
- We expect that analogous phenomena take place in the other mean motion resonances.

**Theorem** For the Restricted Planar Elliptic 3 Body Problem with mass ratio  $\mu = 10^{-3}$  and eccentricity  $e_0$  small enough, there exist  $T > 0$  and a trajectory whose (osculating) semimajor  $a(t)$  and eccentricity  $e(t)$  satisfy that

$$a(t) \sim 7^{2/3} \quad \text{for all } t \in [0, T]$$

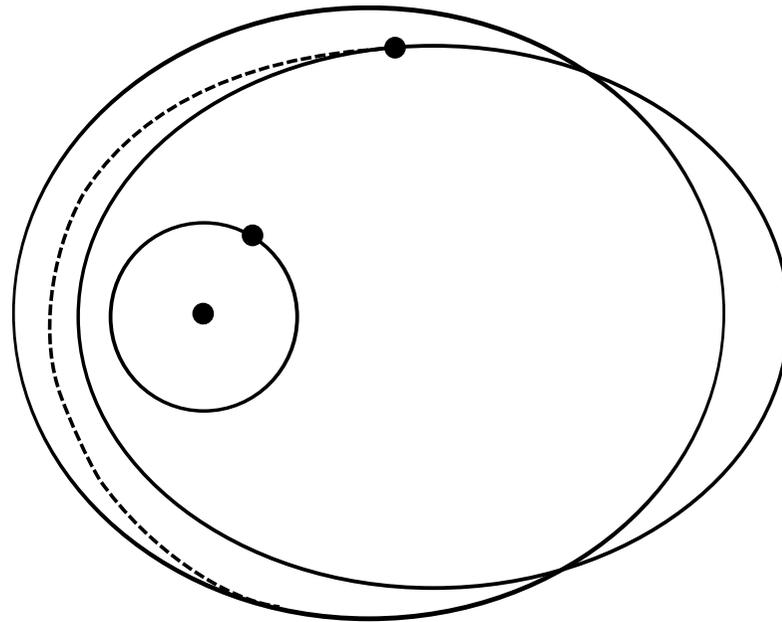
and

$$e(0) < 0.48 \quad \text{and} \quad e(T) > 0.66.$$

Namely,

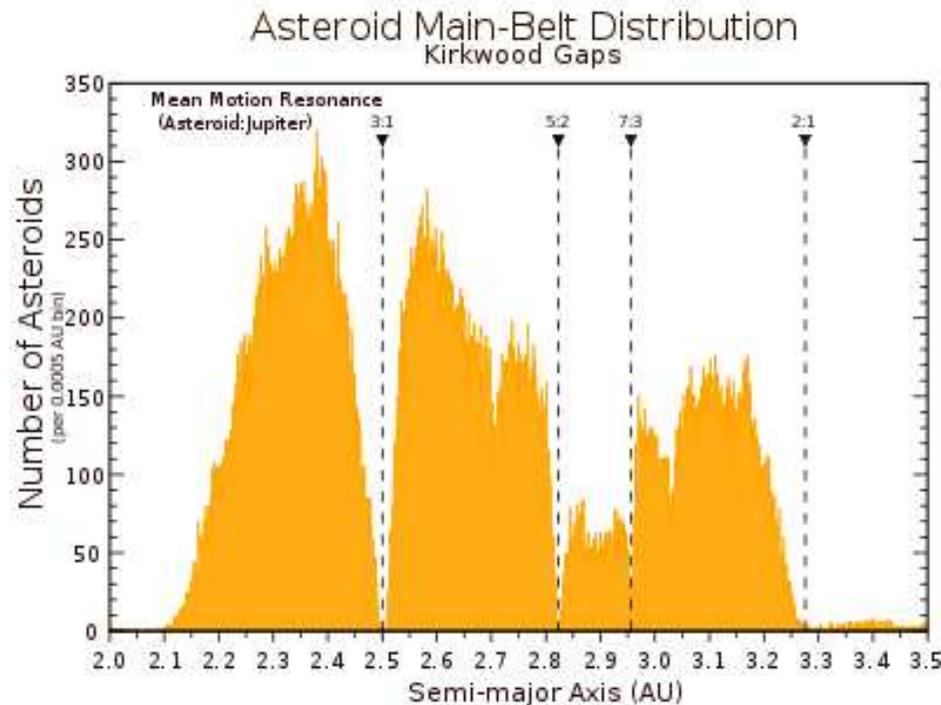
- The Asteroid keeps the semimajor axis almost constant and thus it remains in mean motion resonance.
- It drifts along the resonance, undergoing considerable changes in the eccentricity and thus in the shape of the ellipse.

- Schematically, we obtain orbits:



- Even though we have focused on the resonance  $1 : 7$ , we expect the same to happen in the other mean motion resonances.

## The Kirkwood gaps



- The Asteroid Belt is the region of the Solar System located roughly between the orbits of the planets Mars and Jupiter.

- At mean motion resonances of small order 3 : 1, 2 : 1, 5 : 2, 7 : 3, there are visible gaps in the distribution of the Asteroids, called Kirkwood gaps.

- This diffusing mechanism could give a justification of its existence.
- The eccentricity of Jupiter is  $e_0 \sim 1/20$  and we need  $e_0$  arbitrarily small.
- Another mechanism of instability in the 3 : 1 Kirkwood gap based on Adiabatic chaos can be seen in Neishtadt-Sidorenko (2004).

## General comments on the proof

- The proof relies on geometric methods commonly used in the study of Arnol'd diffusion.
- This problem does **not** have **big gaps**.
- Some parts rely on (rather simple) numerical computations.
- We expect that these parts can be turned into a Computer Assisted Proof.

## Sketch of the proof

Main steps:

- Step 1: Consider the Action-Angle coordinates for the 2BP in elliptic regime (Delaunay coordinates)
- Step 2: Geometrical features of the Circular Problem ( $\mu = 10^{-3}$  and  $e_0 = 0$ ).
- Step 3: Study of the Elliptic Problem ( $e_0 > 0$ ) as a perturbation of the Circular One.

## The two body problem

- When  $\mu = 0$ , the Hamiltonian becomes

$$H(q, p) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|}.$$

- The Delaunay coordinates are the Action-Angle coordinates for the 2BP in elliptic regime:
  - $\ell$  is the mean anomaly.
  - $L$  is the square of the semimajor axis.
  - $\tilde{g}$  is the argument of the perihelion.
  - $G$  is the angular momentum.
- One can define the eccentricity of the Asteroid using these coordinates as

$$e = \sqrt{1 - \frac{G^2}{L^2}}.$$

- In these coordinates the Hamiltonian of the 2BP become

$$H_{2BP}(\ell, L, \tilde{g}, G) = -\frac{1}{2L^2}.$$

- If we apply the change to the RPE3BP we obtain

$$H(\ell, L, \tilde{g}, G) = -\frac{1}{2L^2} + \mu\Delta H_{\text{circ}}(\ell, L, \tilde{g} - t, G) \\ + \mu e_0 \Delta H_{\text{ell}}(\ell, L, \tilde{g} - t, G, t)$$

- The circular perturbing term only depends on  $t$  through  $\tilde{g} - t$ .

## Rotating Delaunay coordinates

- One can define a new system of coordinates with  $g = \tilde{g} - t$ : **Rotating Delaunay coordinates**.
- New Hamiltonian

$$H(\ell, L, g, G, t) = -\frac{1}{2L^2} - G + \mu\Delta H_{\text{circ}}(\ell, L, g, G) + \mu e_0 \Delta H_{\text{ell}}(\ell, L, g, G, t)$$

- Since  $\Delta H_{\text{circ}}$  is independent of  $t$ , when  $e_0 = 0$  the system has 2 degrees of freedom and the energy is preserved.
- This corresponds to the preservation of the Jacobi constant.
- We will look for diffusing orbits in this Hamiltonian when  $e_0 > 0$ .

## The mean motion resonance in Rotating Delaunay coordinates

- When  $\mu = 0$  we have the Hamiltonian

$$H(\ell, L, g, G, ) = -\frac{1}{2L^2} - G$$

- Frequencies:

$$\dot{\ell} = \frac{1}{L^3} \quad \text{and} \quad \dot{g} = -1$$

- The mean motion resonance 1 : 7 corresponds to take  $L = 7^{1/3}$  so that

$$\dot{\ell} = \frac{1}{7} \quad \text{and} \quad \dot{g} = -1$$

- We want to obtain diffusing orbits along the mean motion resonance.
- Namely we will keep  $L \sim 7^{1/3}$  (which implies keeping the semimajor axis almost constant)
- Since

$$e = \sqrt{1 - \frac{G^2}{L^2}},$$

big changes in  $G$  are equivalent to big changes in  $e$ .

## The extended system

- In fact, we will consider the full 3 dof freedom system introducing the variable  $I$  conjugate of  $t$

$$H(\ell, L, g, G, t, I) = -\frac{1}{2L^2} - G + \mu\Delta H_{\text{circ}}(\ell, L, g, G) \\ + \mu e_0 \Delta H_{\text{ell}}(\ell, L, g, G, t) + I$$

- We restrict ourselves at the energy level  $H = 0$ .
- Since the perturbing terms are small and  $L$  almost constant, to obtain an orbit with big changes in  $G$  is equivalent to obtain an orbit with big changes in  $I$ .
- We look for orbits with big changes in  $I$ .

## The circular problem

- For a moment let us forget about  $I$  and  $t$ .
- The Hamiltonian for the circular problem is

$$H(\ell, L, g, G) = -\frac{1}{2L^2} - G + \mu\Delta H_{\text{circ}}(\ell, L, g, G)$$

- The energy is conserved.
- It has two degrees of freedom so it is impossible to obtain diffusion.
- We study the mean motion resonance numerically.
- We take advantage of the fact that this system is reversible with respect to the involution

$$R(\ell, L, g, G) = (-\ell, L, -g, G).$$

**Theorem** Consider the Hamiltonian

$$H(\ell, L, g, G) = -\frac{1}{2L^2} - G + \mu\Delta H_{\text{circ}}(\ell, L, g, G)$$

Then, at each energy level  $H \in [H_-, H_+] = [-1.81, -1.56]$ ,

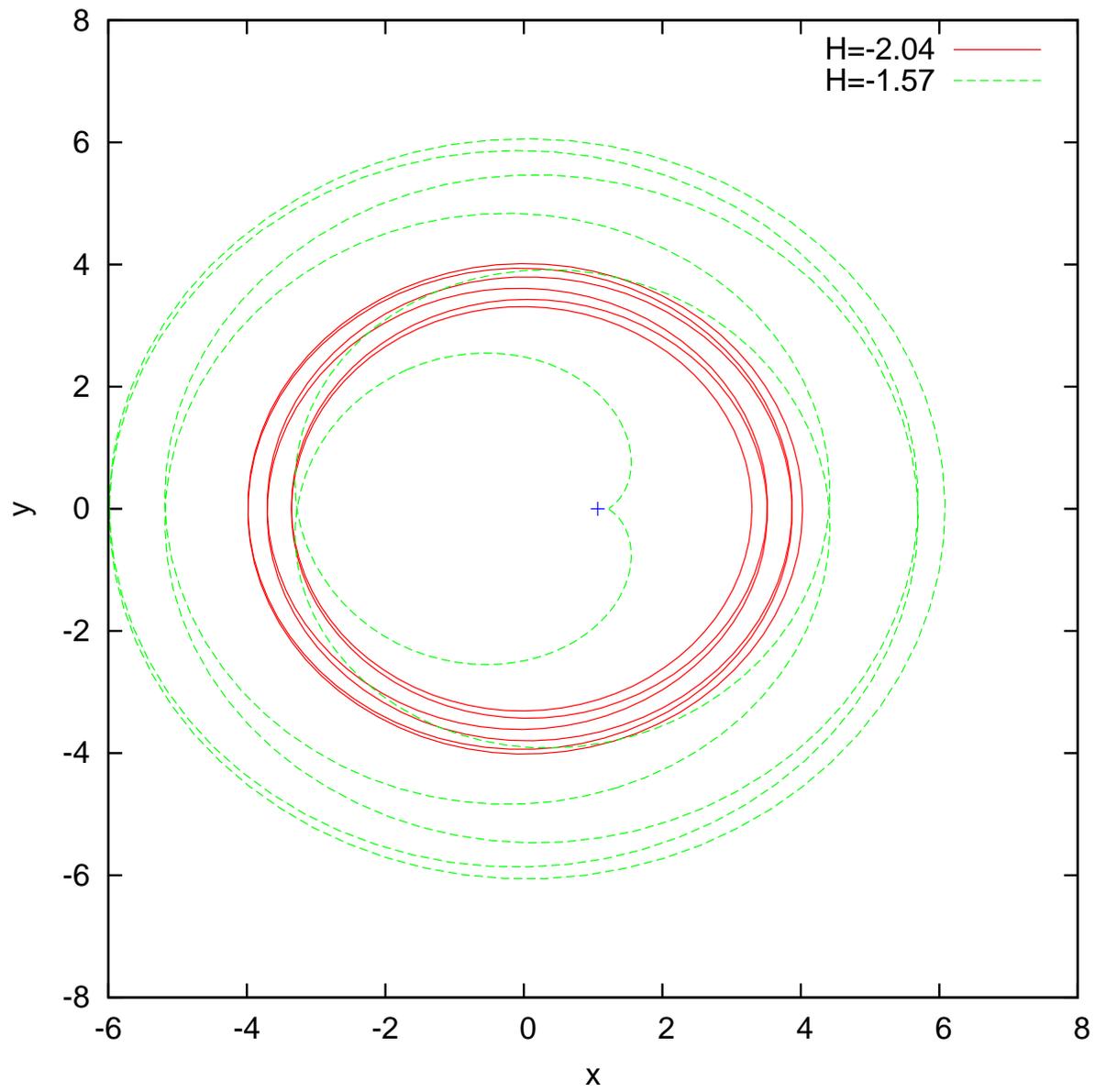
- There exists a hyperbolic periodic  $\gamma_H$  orbit whose period satisfies

$$|T - 14\pi| < 60\mu = 60 \cdot 10^{-3}$$

- $\gamma_H$  has two branches of stable and unstable invariant manifolds.
- At each energy level either one set of branches of  $\gamma_H$  or the other intersect transversally at the symmetry axis.

## Remarks on the theorem

- $H \rightarrow H_-$  implies that  $e$  decreases (the orbit of the Asteroid becomes more circular).
- In this regime the periodic orbit becomes weakly hyperbolic and the angle between the invariant manifolds decreases exponentially.
- It becomes harder to detect.
- When  $H \rightarrow H_+$  the periodic orbit approaches the invariant manifolds of the point Lagrangian Equilibrium Point  $L_2$ .
- The period of the periodic orbit explodes and thus we move away from the resonance.



## A priori chaotic versus a priori stable

- This theorem gives us at every energy level a periodic orbit which has a transversal homoclinic orbit for  $\mu = 10^{-3}$ .
- We will use this hyperbolic structure to obtain diffusing orbits when  $e_0 > 0$ .
- This type of systems are usually called *a priori chaotic*, since for the unperturbed problem ( $e_0 = 0$ ) they present chaotic motion at each energy level (but not global instabilities).
- It presents similar features to the so-called Mather Problem: existence of orbits whose energy grows arbitrarily in geodesic flows with a periodic potential.

- When  $\mu \rightarrow 0$ , the system becomes nearly completely integrable (unperturbed problem only depends on the actions): it is an *a priori stable* system.
- The splitting angle at the homoclinic points is exponentially small with respect to  $\mu$  and therefore it is very difficult to prove the transversality of the invariant manifolds.

## The extended circular problem

- Hamiltonian for the extended circular problem

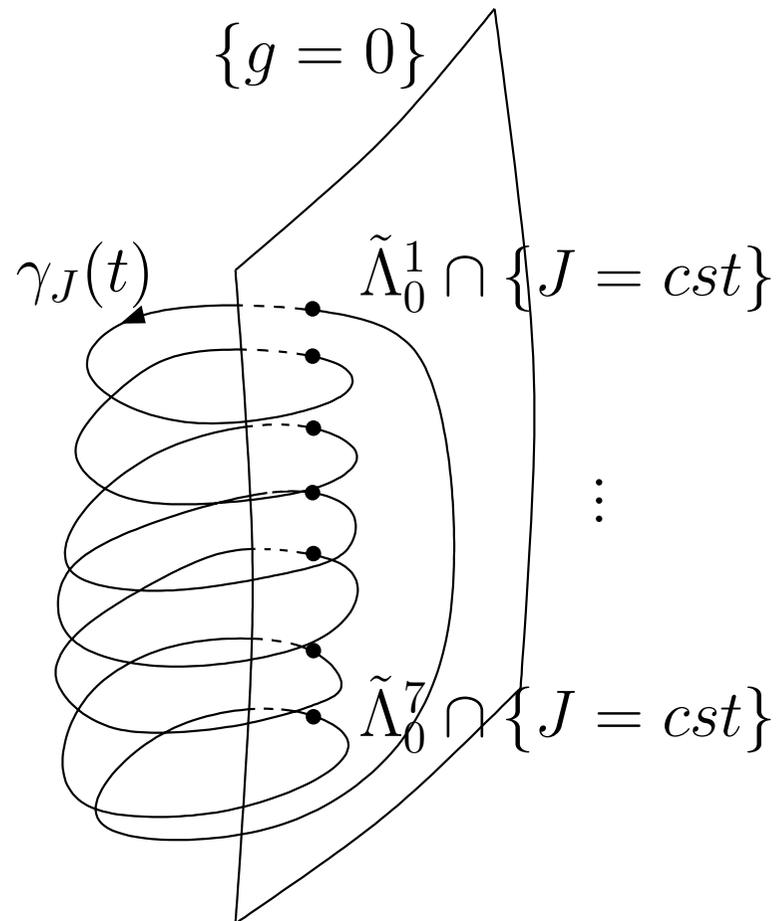
$$H(\ell, L, g, G, t, I) = -\frac{1}{2L^2} - G + \mu\Delta H_{\text{circ}}(\ell, L, g, G) + I$$

at the energy level  $H = 0$ .

- Now the dynamics is restricted to planes  $I = \text{constant}$ .
- If we take into account the variable  $t$ , the periodic orbits of the circular problem are now 2 dimensional tori
- The union of these 2 dimensional tori form a 3 dimensional Normally Hyperbolic Invariant Manifold  $\Lambda_0$ .
- We want to define inner and outer dynamics associated to it.

- To this end we consider a Poincaré map.
- We fix the Poincaré section  $\{g = 0\}$  and the map

$$\mathcal{P}_0 : \{g = 0\} \longrightarrow \{g = 0\}$$



- The Poincaré map  $\mathcal{P}_0$  has a 2 dimensional NHIM  $\tilde{\Lambda}_0$
- It has seven connected components

$$\tilde{\Lambda}_0 = \cup_{j=0}^6 \tilde{\Lambda}_0^j$$

- In fact,  $\mathcal{P}_0(\tilde{\Lambda}_0^j) = \tilde{\Lambda}_0^{j+1}$ .
- They are invariant by  $\mathcal{P}_0^7$ .
- $\mathcal{P}_0^7$  has seven NHIMs:

$$\tilde{\Lambda}_0^j, j = 0, \dots, 6.$$

- $(I, t)$  are global coordinates for each of these connected components.
- We can use them to define the inner and outer dynamics.
- One could also have used the Poincaré map associated to  $\{t = 0\}$  and use coordinates  $(G, g)$ .
- The advantage of using  $(I, t)$  is that  $I$  is constant for the circular problem and therefore it will be easier to study the influence of the elliptic perturbation in order to prove diffusion.

## Inner and outer dynamics of the circular problem

- We chose one of the cylinders:  $\tilde{\Lambda}_0^3$ .
- Recall that it is invariant by  $\mathcal{P}_0^7$ .
- By  $\mathcal{P}_0^7$  it has heteroclinic connections with  $\tilde{\Lambda}_0^2$  and  $\tilde{\Lambda}_0^4$ .
- We choose it because the heteroclinic connections between  $\tilde{\Lambda}_0^3$  and  $\tilde{\Lambda}_0^4$  intersect transversally at the symmetry axis and thus are easier to compute.
- We want to define:
  - Inner dynamics
  - Outer dynamics

## Inner map of the circular problem

- It is given by the Poincaré map  $\mathcal{P}_0^7$  restricted to  $\tilde{\Lambda}_0^3$ .
- Since  $I$  is constant, it is integrable.
- Then, it is of the form

$$\mathcal{F}_0^{\text{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mathcal{T}_0(I) \end{pmatrix}.$$

- $14\pi + \mathcal{T}_0(I)$  is the period of the periodic orbit we have obtained for the circular problem.
- It can be checked (numerically) that it is twist.

## Outer map of the circular problem

- At each energy level either  $W^u(\tilde{\Lambda}_0^3) \pitchfork W^s(\tilde{\Lambda}_0^4)$  or  $W^u(\tilde{\Lambda}_0^4) \pitchfork W^s(\tilde{\Lambda}_0^3)$ .
- Associated to the transversal homoclinic points we can define a scattering map, which we call forward or backward.

$$\mathcal{S}_0^f : \tilde{\Lambda}_0^3 \rightarrow \tilde{\Lambda}_0^4$$

$$\mathcal{S}_0^b : \tilde{\Lambda}_0^4 \rightarrow \tilde{\Lambda}_0^3$$

- $\mathcal{S}_0^*(x_-) = x_+$  if the homoclinic orbit tends to  $x_-$  in the past and to  $x_+$  in the future.

- We want to define outer maps from  $\tilde{\Lambda}_0^3$  to itself  $\tilde{\Lambda}_0^3$ .
- We compose the scattering map with the Poincaré map

$$\mathcal{F}_0^{\text{out},f} = \mathcal{P}_0^6 \circ \mathcal{S}_0^f$$

$$\mathcal{F}_0^{\text{out},b} = \mathcal{S}^b \circ \mathcal{P}_0$$

- As  $I$  is a first integral, the outer maps (wherever they are defined) are of the form

$$\mathcal{F}_0^{\text{in},*} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \omega^*(I) \end{pmatrix} \quad * = f, b$$

- We compute  $\omega^*(I)$  numerically.

## Conclusion

- We have inner and (two) outer dynamics associated to  $\tilde{\Lambda}_0^3$  for the circular problem.
- They are given by

$$\mathcal{F}_0^{\text{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mathcal{T}_0(I) \end{pmatrix}$$

and

$$\mathcal{F}_0^{\text{out},*} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \omega^*(I) \end{pmatrix} \quad * = \text{f, b}$$

- They are all integrable.

## The elliptic problem

- We study the elliptic problem ( $e_0 > 0$ ) as a perturbation of the circular one ( $e_0 = 0$ ).
- For  $e_0$  small enough
  - The NHIM  $\tilde{\Lambda}_0^j$  are preserved, slightly deformed, as  $\tilde{\Lambda}_{e_0}^j$ .
  - Roughly speaking, for each  $I$ ,  $W^u(\tilde{\Lambda}_0^3) \pitchfork W^s(\tilde{\Lambda}_0^4)$  or  $W^u(\tilde{\Lambda}_0^4) \pitchfork W^s(\tilde{\Lambda}_0^3)$  are transversal.
  - We can associate inner and outer dynamics to  $\tilde{\Lambda}_0^3$ :  $\mathcal{F}_{e_0}^{\text{in}}$ ,  $\mathcal{F}_{e_0}^{\text{out},f}$  and  $\mathcal{F}_{e_0}^{\text{out},b}$  as in the circular problem.
  - We study them perturbatively.

## A particular feature of the elliptic Hamiltonian

- Hamiltonian

$$H(\ell, L, g, G, t, I) = -\frac{1}{2L^2} - G + \mu\Delta H_{\text{circ}}(\ell, L, g, G) \\ + \mu e_0 \Delta H_{\text{ell}}(\ell, L, g, G, t) + I$$

- We can expand

$$\Delta H_{\text{ell}} = \Delta H_{\text{ell}}^1 + e_0 \Delta H_{\text{ell}}^2 + \mathcal{O}(e_0^2)$$

- $\Delta H_{\text{ell}}^1$  only has the  $t$ -harmonics  $\{\pm 1\}$ .
- $\Delta H_{\text{ell}}^2$  only has the  $t$ -harmonics  $\{0, \pm 1, \pm 2\}$ .

## The perturbed inner map

- Is of the form

$$\mathcal{F}_{e_0}^{\text{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 A_1(I, t) + e_0^2 A_2(I, t) + \mathcal{O}(e_0^3) \\ t + \mathcal{T}_0(I) + e_0 \mathcal{T}_1(I, t) + e_0^2 \mathcal{T}_2(I, t) + \mathcal{O}(e_0^3) \end{pmatrix}.$$

- $A_1$  and  $\mathcal{T}_1$  only have  $t$ -harmonics  $\{\pm 1\}$ .
- $A_2$  and  $\mathcal{T}_2$  only have  $t$ -harmonics  $\{0, \pm 1, \pm 2\}$ .
- As we will see later, we only need to know explicitly

$$A_1(I, t) = A_1^+(I)e^{it} + A_1^-(I)e^{-it}.$$

- We compute  $A_1^\pm(I)$  numerically.

## The perturbed outer maps

- Are of the form

$$\mathcal{F}_{e_0}^{\text{out},*} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 B_1^*(I, t) + \mathcal{O}(e_0^2) \\ t + \omega^*(I) + \mathcal{O}(e_0) \end{pmatrix} \quad * = \text{f, b.}$$

- $B_1^*$  are computed (numerically) through Melnikov integrals.
- $B_1^*$  only has  $t$ -harmonics  $\{\pm 1\}$ .

- We have defined inner and outer dynamics.
- We want to combine them to obtain a transition chain of tori.
- Namely, we want to obtain a sequence of invariant tori of the inner map  $\{\mathbb{T}_j\}_{j=1\dots N}$  such that

$$W^u(\mathbb{T}_j) \cap W^s(\mathbb{T}_{j+1})$$

- It is equivalent to see that for each tori  $\mathbb{T}_j$  either

$$\mathcal{F}_{e_0}^{\text{out},f}(\mathbb{T}_j) \cap \mathbb{T}_{j+1} \neq \emptyset \quad \text{or} \quad \mathcal{F}_{e_0}^{\text{out},b}(\mathbb{T}_j) \cap \mathbb{T}_{j+1} \neq \emptyset$$

- To obtain it we perform two steps of averaging to the inner map.

## Two steps of averaging

- Inner map:

$$\mathcal{F}_{e_0}^{\text{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 A_1(I, t) + e_0^2 A_2(I, t) + \mathcal{O}(e_0^3) \\ t + \mathcal{T}_0(I) + e_0 \mathcal{T}_1(I, t) + e_0^2 \mathcal{T}_2(I, t) + \mathcal{O}(e_0^3) \end{pmatrix}.$$

- The  $e_0$  and  $e_0^2$  terms of the inner map only have  $t$ -harmonics  $\{0, \pm 1, \pm 2\}$ .
- If we perform two steps of averaging, the small divisors that appear are only

$$e^{\pm i \mathcal{T}_0(I)} - 1 \quad \text{and} \quad e^{\pm 2i \mathcal{T}_0(I)} - 1$$

- The bounds we know for  $\mathcal{T}_0(I)$  show that these small divisors never vanish.

## The inner map in the new variables

- Then, we can perform the two steps of averaging globally.
- Inner map in the new variables:

$$\mathcal{F}_{e_0}^{\text{in}} : \begin{pmatrix} \mathcal{I} \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{I} + \mathcal{O}(e_0^3) \\ \tau + \mathcal{T}_0(\mathcal{I}) + e_0^2 \mathcal{T}_2(\mathcal{I}) + \mathcal{O}(e_0^3) \end{pmatrix}.$$

- The new inner map is  $e_0^3$ -close to integrable.
- It is a twist map.
- We can apply KAM Theorem.
- We obtain a sequence of tori  $\{\mathbb{T}_j\}_{j=1,\dots,N}$ , which are  $e_0^{3/2}$ -close to each other.

## The outer map in the new variables

- To obtain the transition chain, we consider the outer maps in the new variables:

$$\tilde{\mathcal{F}}_{e_0}^{\text{out},*} : \begin{pmatrix} \mathcal{I} \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{I} + e_0 \tilde{B}^*(\mathcal{I}, \tau) + \mathcal{O}(e_0^2) \\ \tau + \omega^*(\mathcal{I}) + \mathcal{O}(e_0) \end{pmatrix}, \quad * = \text{f, b}$$

where

$$\tilde{B}^*(\mathcal{I}, \tau) = \tilde{B}^{*,+}(\mathcal{I}) e^{i\tau} + \tilde{B}^{*,-}(\mathcal{I}) e^{-i\tau}$$

with

$$\tilde{B}^{*,\pm}(\mathcal{I}) = B^{*,\pm}(\mathcal{I}) - \frac{e^{\pm i\omega^*(\mathcal{I})} - 1}{e^{\pm i\mathcal{T}_0(\mathcal{I})} - 1} A_1^{\pm}(\mathcal{I}).$$

- We want the outer map to connect tori which are  $e_0^{3/2}$ -close.
- It is enough to check that  $\tilde{B}^{*,\pm}(\mathcal{I}) \neq 0$ .
- Namely,  $\tilde{B}^{*,\pm}$  is defined through the functions  $A_1^{\pm}$ ,  $B^{*,\pm}$ ,  $\mathcal{T}_0$  and  $\omega^*$ , which have been already computed.

- Since  $\tilde{B}^{*,\pm}(\mathcal{I}) \neq 0$ , the jumps of the outer maps are bigger than the distance between tori.
- Namely, there are no **big gaps**.
- Therefore,  $\{\mathbb{T}_j\}_{j=1,\dots,N}$  is a transition chain.
- Once we have the transition chain, it is enough to use a shadowing method to obtain the true orbit.

## Open questions

- Obtain diffusing orbits all the way until eccentricities of the Asteroid  $e \sim 0$ , namely up to a neighborhood of circular motions of the Asteriod.
- Prove diffusion for  $\mu \rightarrow 0$  (one needs to face exponentially small phenomena).

## Time of diffusion

- The used methods do not give any estimate on the time of diffusion.
- We expect that, for the intermediate range in  $e$ 's for which we obtain diffusion, it is of the form

$$T \sim \frac{\ln(\mu e_0)}{\mu^{3/2} e_0}.$$

- Different diffusing orbits (which could be obtained using different methods) may need shorter time

$$T \sim \frac{\ln \mu}{\mu^{3/2} e_0}.$$

- We expect this time of diffusion to be true even for  $\mu \rightarrow 0$  (even if we are not able to obtain diffusing orbits for this case).
- To obtain diffusing orbits reaching eccentricities  $e \sim 0$  probably may require longer time.